q-expansion principles for modular curves at infinite level

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Abstract

We develop an analytic theory of cusps of the modular curve at infinite level $\mathcal{X}^*_{\Gamma(p^\infty)}$ and some lower level modular curves in terms of perfectoid parameter spaces for Tate curves. We then prove various q-expansions principles for functions on perfectoid modular curves, namely that the properties of extending to the cusps, vanishing, coming from finite level, and being bounded, can all be detected on q-expansions. As an application, we show that there is a canonical tilting isomorphism $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a = \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$.

Contents

1 Introduction

Let K be a perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$ and let $\mathcal{X}^*$ be the modular curve of some tame level $\Gamma(N)$ over K, considered as an analytic adic space. In the first part of this paper, we carry out a detailed analysis of the geometry near the cusps in the inverse system of modular curves with higher level structures at p. This complements in the case of dimension 1 results of Scholze on the boundary of Siegel moduli spaces for abelian varieties of dimension ≥ 2 in [11], proved there using machinery like a perfectoid version of Riemann's Hebbarkeitssatz, which due to codimension 2 assumptions only apply for higher dimensional Siegel spaces.

Our way to study the boundary in the elliptic case is to develop a theory of analytic Tate curve parameter spaces: These are moduli spaces of Tate curves, and in the simplest case are rigid open discs $D \subseteq \mathrm{Spa}(K\langle q\rangle)$ of radius 1, defined by the condition $|q| < 1$. For the tame level modular curve $\mathcal{X}^*$, it is a consequence of a Theorem by Conrad [3] that for any cusp c of $\mathcal{X}^*$, there is a canonical open immersion $D \hookrightarrow \mathcal{X}^*$ that sends the origin to that cusps. It then follows essentially from the classical calculus of cusps after Katz–Mazur [9] that for any cusp of the tame level modular curve $\mathcal{X}^*$ there are Cartesian diagrams

$$\begin{array}{ccccccc}
\underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D & \longrightarrow & \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D & \longrightarrow & D & \xrightarrow{q \mapsto q^{p^n}} & D \\
\big\downarrow{\scriptstyle\varphi} & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{X}^*_{\Gamma(p^n)}(0)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^n)}(0)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}(0)_a & \longrightarrow & \mathcal{X}^*(0)
\end{array}$$

canonical after a choice of p^n-th root of unity in K.

We show that in the limit $n \to \infty$, these open subspaces give rise to perfectoid Tate parameter spaces given by the perfectoid open discs $D_\infty \subseteq \mathrm{Spa}(K\langle q^{1/p^\infty}\rangle)$ defined by $|q| < 1$. The above diagram then in the limit becomes a Cartesian diagrams of perfectoid spaces

$$\begin{array}{ccccccc}
\underline{\Gamma_0(p^\infty)} \times D_\infty & \longrightarrow & \underline{\mathbb{Z}_p^\times} \times D_\infty & \longrightarrow & D_\infty & \longrightarrow & D \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a & \longrightarrow & X^*_{\Gamma_1(p^\infty)}(0)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a & \longrightarrow & \mathcal{X}^*(0)
\end{array}$$

where $\Gamma_0(p^\infty) := \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)\} \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ and $\underline{\Gamma_0(p^\infty)}$ and $\underline{\mathbb{Z}_p^\times}$ are perfectoid groups which are profinite tilde-limits. By identifying the action of $\Gamma_0(p) := \{\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) | c \in p\mathbb{Z}_p\} \subseteq \mathrm{GL}_2(\mathbb{Z}_p)$ on $\underline{\Gamma_0(p^\infty)} \times D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$, one can deduce via $\mathrm{GL}_2(\mathbb{Z}_p)$-translations the following:

Theorem 1.1. *1. Consider the right action of $\mathbb{Z}_p$ on the perfectoid space $\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty$ defined by $(\gamma, q) \cdot h \mapsto (\gamma \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right), q^{1/p^\infty}\zeta_{p^\infty}^h)$. Then the quotient $(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p$ exists as an adic space. Let c be any cusp of $\mathcal{X}^*$. Then the pullback of the corresponding Tate parameter space $D \hookrightarrow \mathcal{X}^*$ along the projection $\mathcal{X}^*_{\Gamma(p^\infty)} \to \mathcal{X}^*$ is of the form*

$$\begin{array}{ccc}
(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p & \longrightarrow & D \\
\big\downarrow & & \big\downarrow \\
\mathcal{X}^*_{\Gamma(p^\infty)} & \longrightarrow & \mathcal{X}^*
\end{array}$$

where the morphism on top is projection to the second factor. The morphism on the left is canonical after a choice of ζ_{p^∞} and is then $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant for the natural left action on $(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p$ induced by letting $\mathrm{GL}_2(\mathbb{Z}_p)$ act on the first factor.

2. *The map* $\pi_{HT} : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathbb{P}^1$ *restricts to* $(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p \to \underline{\mathbb{P}^1(\mathbb{Z}_p)}$ *given by*

$$(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), q) \mapsto (b : d).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} (\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p & \longrightarrow & \underline{\mathbb{P}^1(\mathbb{Z}_p)} \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^*_{\Gamma(p^\infty)} & \xrightarrow{\quad \pi_{HT} \quad} & \mathbb{P}^1. \end{array}$$

In particular, the Theorem implies that for any cusp c of $\mathcal{X}^*$, the cusps of $\mathcal{X}^*_{\Gamma(p^\infty)} \to \mathcal{X}^*$ over c form a closed profinite subspace $\underline{\mathrm{GL}_2(\mathbb{Z}_p)/\mathbb{Z}_p} \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}$. For each $\gamma \in \mathrm{GL}_2(\mathbb{Z}_p)/\mathbb{Z}_p$, we will denote by c_γ the cusp of $\mathcal{X}^*_{\Gamma(p^\infty)}$ defined by specialising at $\gamma : \mathrm{Spa}(K) \to \underline{\mathrm{GL}_2(\mathbb{Z}_p)/\mathbb{Z}_p}$.

The Tate parameter spaces give a way to talk about q-expansions of functions on modular curves. They can be useful when working with modular curves at infinite level, as they often allow one to extend constructions which are a priori defined only away from the cusps, for instance maps defined by the moduli interpretation, to the compactifications. Explicitly, we for instance have the following immediate consequence:

Corollary 1.1. *Let* $c_0, \dots, c_m$ *be a collection of cusps of* $\mathcal{X}'^*$ *such that each connected component of* $\mathcal{X}^*$ *contains at least one* c_i. *For each* c_i *let* $S(c_i) \subseteq \mathrm{GL}_2(\mathbb{Z}_p)/\mathbb{Z}_p$ *be a dense subset. Then a function* f *on* $\mathcal{X}_{\Gamma(p^\infty)}$ *can be extended to a function on* $\mathcal{X}^*_{\Gamma(p^\infty)}$ *if and only if for all* c_i *and all* $\gamma \in S(c_i)$, *the* q*-expansion of* f *at the cusp* $c_{i,\gamma}$ *is already contained in* $\mathcal{O}_K[[q^{1/p^\infty}]][1/p] \subseteq \mathcal{O}_K((q^{1/p^\infty}))[1/p]$. *In this case, the extension of* f *is unique.*

In the second part of this article we show that, in a similar fashion, one can use the Tate parameter spaces to prove various q-expansion principles which are often useful when working with functions on infinite level modular curves, like modular forms:

Corollary 1.2 (q-expansion principle I: detecting vanishing)**.** *Let* $c_0, \dots, c_m$ *be a collection of cusps of* $\mathcal{X}'^*$ *such that each connected component of* $\mathcal{X}^*$ *contains at least one* c_i. *Then restriction of functions gives injective maps*

$$\begin{aligned} \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a) &\hookrightarrow \prod_{i=1}^m \mathcal{O}_K[[q^{1/p^\infty}]][1/p] \\ \mathcal{O}(\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a) &\hookrightarrow \prod_{i=1}^m \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_K[[q^{1/p^\infty}]])[1/p] \\ \mathcal{O}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a) &\hookrightarrow \prod_{i=1}^m \mathrm{Map}_{\mathrm{cts}}(\Gamma_0(p^\infty), \mathcal{O}_K[[q^{1/p^\infty}]])[1/p]. \end{aligned}$$

Corollary 1.3 (q-expansion principle II: detecting the level)**.** *Let* $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a)$ *be a function on* $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$. *Then for any* $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, *the following are equivalent:*

1. f *is the pullback of a function on* $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$.
2. *The* q*-expansion of* f *at every cusp is already in* $\mathcal{O}_K[[q^{1/p^n}]][1/p] \subseteq \mathcal{O}_K[[q^{1/p^\infty}]][1/p]$.
3. *On every connected component of* $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ *there is at least one cusp at which the* q*-expansion of* f *is already in* $\mathcal{O}_K[[q^{1/p^n}]][1/p] \subseteq \mathcal{O}_K[[q^{1/p^\infty}]][1/p]$.

Proposition 1.4 (q-expansion principle III: detecting boundedness on the ordinary locus)**.** *A function $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma_1(p^\infty)}(0)) = S$ is contained in $S^\circ = \mathcal{O}^+(\mathcal{X}^*_{\Gamma_1(p^\infty)}(0))$ if and only if its q-expansion is in $\mathcal{O}_K[[q^{1/p^\infty}]]$. The analogous statement for $\mathcal{X}'^*_{Ig(p^\infty)}(0)^{\mathrm{perf}} = \mathrm{Spa}(S^\flat, S^{\flat\circ})$ is also true: An element of $S^\flat$ is in $S^{\flat\circ}$ if and only if its q-expansion is in $\mathcal{O}_{K^\flat}[[q^{1/p^\infty}]]$.*

Finally, we give an example of an application of Tate parameter spaces and the q-expansion principle of extending to the cusps, which we are interested in for applications to modular forms: It is an extension of a result from [11], III.2.5, proved there for the Siegel space paramatrising abelian varieties of dimension $g \geq 2$, to the case $g = 1$ of elliptic curves:

Theorem 1.2. *1. There is a canonical isomorphism $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a \xrightarrow{\sim} \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ which is $\mathbb{Z}_p^\times$-equivariant and makes the following diagram commute:*

$$\begin{array}{ccc} \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a & \xrightarrow{\sim} & \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \\ \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a & \xrightarrow{\sim} & \mathcal{X}'^*(\epsilon)^{\mathrm{perf}} \end{array}$$

where the isomorphism on the bottom line is the one from [11], Corollary III.2.19.

2. The cusps of $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ and $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)$ correspond via the isomorphism in 1. For any pair of cusps, the corresponding Tate parameter spaces fit into a commutative diagram

$$\begin{array}{ccc} \underline{\mathbb{Z}_p^\times} \times D^\flat_\infty & \hookrightarrow & \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a \\ \| & & \| \\ \underline{\mathbb{Z}_p^\times} \times D'_\infty & \hookrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \end{array}$$

where D' is the open rigid unit disc over $K^\flat$ and D'_∞ is its perfection.

A third application of the Tate parameter spaces at infinite level can be found in [2].

We remark that this article is a prequel to the paper "tilting equivalences of modular forms" [6] about a perfectoid perspective on modular forms at the boundary of weight space.

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2 Tate parameter spaces for finite level modular curves

Throughout let p be a prime. Let K be a p-adic perfectoid field extension of $\mathbb{Q}_p^{\mathrm{cyc}}$.

Let us briefly recall some notation from [11] for adic and perfectoid modular curves: Let N be some integer ≥ 5 coprime to p. Let X be the modular curve of some tame level Γ^p at N over K and let X^* denote its compactification. For simplicity we shall always assume that K contains a primitive N-root of unity, so that X and X^* both decompose into $\varphi(N)$

disjoint irreducible components. When in the following we consider the modular curves over a ring of definition R other than K, we shall denote it by X_R, and similarly for the compactification. In particular, there are the integral models $X_{\mathcal{O}_K}$ and $X^*_{\mathcal{O}_K}$. We denote by $\mathfrak{X}$ and $\mathfrak{X}^*$ their respective p-adic completions. We use calligraphic letters to denote the adic analytifications $\mathcal{X}$ and $\mathcal{X}^*$ of X_K and $\mathcal{X}^*$ their adic generic fibres – this is the only way in which we deviate from the notation in [11], where $\mathcal{X}$ denoted the good reduction locus, which we shall denote by $\mathcal{Y}$.

For any of the classical Katz-Mazur level structures $\Gamma = \Gamma_0(p^n), \Gamma_1(p^n), \Gamma(p^n)$, $n \in \mathbb{N}$, we denote by $X_\Gamma \to X$ the representing moduli scheme. Similarly, there are $X^*_\Gamma \to X^*$, and adic space $\mathcal{X}_\Gamma \to X_\Gamma$, $\mathcal{X}^*_\Gamma$ as well as $\mathcal{Y}_\Gamma \to \mathcal{Y}$. We remark that these spaces have a moduli interpretation in the adic category:

Lemma 2.1. *Let S be an honest adic space over $(K, \mathcal{O}_K)$. Then*

$$\mathrm{Hom}_{\mathbf{Adic}}(S, \mathcal{X}_\Gamma) = X(\mathcal{O}_S(S)).$$

In particular, the S-points of $\mathcal{X}_\Gamma$ are in functorial correspondence with isomorphism classes of elliptic curves over $\mathcal{O}_S(S)$ with tame level structure Γ^p and level structure Γ at p.

Proof. The moduli scheme X_Γ over K is an affine curve ([9] Corollary 4.7.2), say $X_\Gamma = \mathrm{Spec}(A)$. By the universal property of the analytification $\mathcal{Y}_\Gamma = X^{an}_\Gamma$ we then have

$$\mathrm{Hom}_{\mathbf{Adic}}(S, X^{an}_\Gamma) = \mathrm{Hom}_{\mathbf{LRS}}(S, X_\Gamma) = X_\Gamma(\mathcal{O}_S(S))$$

where the last step is the adjunction of Spec and global sections for locally ringed spaces. □

Using local lifts Ha of the Hasse invariant one defines an open subspace $\mathcal{X}^*(\epsilon) \subseteq \mathcal{X}^*$ cut out by the condition that $|\,\mathrm{Ha}\,| \geq |p|^\epsilon$. Following [11], there is a canonical integral model $\mathfrak{X}^*(\epsilon) \to \mathfrak{X}^*$. As a general means of notation, for any adic space $S \to \mathcal{X}^*$ we shall write

$$S(\epsilon) := S \times_{\mathcal{X}^*} \mathcal{X}^*(\epsilon)$$

for the open subspace of S that is the preimage of $\mathcal{X}^*(\epsilon)$, and similarly for the integral models. In particular, for any of the classical Katz-Mazur level structures $\Gamma = \Gamma_0(p^n), \Gamma_1(p^n), \Gamma(p^n)$ the modular curve $\mathcal{X}^*_\Gamma \to \mathcal{X}^*$ restricts to a morphism $\mathcal{X}^*_\Gamma(\epsilon) \to \mathcal{X}^*(\epsilon)$. We note that the open subspace $\mathcal{X}^*(0)$ is precisely the ordinary locus of $\mathcal{X}^*$ (ie the locus of good ordinary or semistable reduction). We therefore say for the elliptic curve represented by $\mathcal{Y}(\epsilon)$ that they are ϵ-nearly ordinary.

By the theory of the canonical subgroup, the forgetful morphism $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon) \to \mathcal{X}^*(\epsilon)$ has a canonical section. We denote by $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_c$ the image of this section, that is the component of $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)$ that parametrises the $\Gamma_0(p)$-structure given by the canonical subgroup. This is called the canonical locus. We denote its complement by $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$ and call it the anticanonical locus. For any adic space $S \to \mathcal{X}^*_{\Gamma_0(p)}$ we denote by

$$S(\epsilon)_a := S \times_{\mathcal{X}^*_{\Gamma_0(p)}} \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$$

the open subspace that lies over the anticanonical locus. For any adic space S with an elliptic curve E over $\mathcal{O}_S(S)$, we shall call the data of a Γ-level structure that corresponds to a point of $\mathcal{Y}_\Gamma(\epsilon)_a$ a Γ_a-level structure. For instance, a $\Gamma_0(p^n)_a$-level structure is the data of a locally free subgroup scheme $D_n \subseteq E[p^n]$ that is fppf-locally cyclic of rank p^n.

Finally in this section, we recall that for any $n \in \mathbb{N}$, the transformation of moduli functors that sends an elliptic curve E together with an $\Gamma(p^n)_a$-structure D_n to the elliptic curve E/D_n induces an isomorphism

$$\mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a \xrightarrow{\sim} \mathcal{X}(p^{-n}\epsilon)$$

that is called the Atkin-Lehner isomorphism. The inverse is given by sending E with canonical subgroup C_n to the data of E/D_n with $\Gamma_0(p^n)_a$-structure $E[p^n]/C_n$. The Atkin-Lehner isomorphism uniquely extends to the cusps for all n, and for varying n the resulting isomorphisms fit into a commutative diagram of towers

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^2)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*(\epsilon) \\ & \wr\Big| & & \wr\Big| & & \Big\| \\ \cdots \longrightarrow & \mathcal{X}^*(p^{-2}\epsilon) & \xrightarrow{F} & \mathcal{X}^*(p^{-1}\epsilon) & \xrightarrow{F} & \mathcal{X}^*(\epsilon) \end{array}$$

where in the bottom row, the morphism F is the "Frobenius lift" defined in terms of moduli by sending E to E/C_1. The resulting tower from above is called the "anticanonical tower".

It is a crucial result of [11] that the anticanonical tower becomes perfectoid in the inverse limit: More precisely:

Theorem 2.1. *There is an affinoid perfectoid space* $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ *such that*

$$\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \sim \varprojlim_{n\in\mathbb{N}} \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$$

Since the forgetful morphisms $\mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ are finite étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsors, even over the cusps, one immediately obtains in the inverse limit an affinoid perfectoid space $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a \sim \varprojlim_n \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$ together with a forgetful map $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ that is a pro-étale $\mathbb{Z}_p^\times$ torsor. Similarly, for full level $\Gamma(p^n)$, one obtains an affinoid perfectoid space $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ together with a forgetful map that is a pro-étale $\Gamma_0(p^\infty)$-torsor $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ where we set:

Definition 2.2. For any $m \in \mathbb{N} \cup \{\infty\}$, let $\Gamma_0(p^m) = \{\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_p) \mid c \equiv 0 \bmod p^m\}$.

All in all, we have a tower of morphism

$$\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \longrightarrow \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a \longrightarrow \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \longrightarrow \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$$

which is a pro-étale $\Gamma_0(p)$-torsor away from the boundary, but not globally since there is ramification over the cusps in $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$.

We also recall that at infinite level, there is the Hodge-Tate period map

$$\pi_{HT} : \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \to \mathbb{P}^1,$$

which is the restriction of the Hodge-Tate period map to the anticanonical locus. In terms of moduli (away from the cusps) this sends any point corresponding to an elliptic curve E over an algebraically closed field C with a trivialisation $\alpha : \mathbb{Z}_p^2 \to T_pE$ to the quotient of C^2 given by the Hodge-Tate map $T_pE \otimes C \to \omega_E$ via the identification $C^2 \cong T_pE \otimes C$ induced by α. This moduli description on points "geometrises" to a natural isomorphism of line bundles $\pi_{HT}^*\mathcal{O}(1) = q^*\omega$ over $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ where ω is the usual automorphic bundle on $\mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a$.

2.1 Analytic Tate parameter spaces and cusps

In this section we recall the theory of the universal rigid analytic Tate curve around the cusp, as developed by [3]. The only two differences are that we use $\Gamma_1(N)$ instead of $\Gamma(N)$, and that we work with analytic adic spaces. In particular, instead of the generalisation of Berthelot's functor constructed in §3 of *loc. cit.* we may use the adic generic fibre functor.

Throughout let $n \in \mathbb{Z}_{\geq 0}$. We consider the modular curve $\mathcal{X}^*_{\Gamma_0(p^n)}$. Note that this includes the case of the tame level modular curve $\mathcal{X}^* = \mathcal{X}^*_{\Gamma_0(1)}$, a case we are also interested in. Recall that we assume that K is a perfectoid field that contains all N-th unit roots, and therefore the cusps of $X^*_{\mathcal{O}_K,\Gamma_0(p^n)}$ are a disjoint union of $\mathcal{O}_K$-points. Let $c : \mathrm{Spec}(\mathcal{O}_K, \Gamma_0(p^n)) \to X^*_{\mathcal{O}_K,\Gamma_0(p^n)}$ be any of the cusps, then completion along the cusp results in a map

$$c : \mathrm{Spf}(\mathcal{O}_K[[q]]) \to X^*_{\mathcal{O}_K,\Gamma_0(p^n)}$$

where $\mathcal{O}_K[[q]]$ is endowed with the q-adic topology. Upon π-adic completion, this gives rise to a morphism

$$\hat{c} : \mathrm{Spf}(\mathcal{O}_K[[q]]) \to \mathfrak{X}^*_{\Gamma_0(p^n)}$$

where now $\mathcal{O}_K[[q]]$ is endowed with the (p, q)-adic topology. We note that this morphism restricts to $\mathrm{Spf}(\mathcal{O}_K[[q]]) \to \mathfrak{X}^*(0)$ since the supersingular locus is disjoint from the cusps.

On the p-adic generic fibre, we obtain a morphism of analytic adic spaces over K

$$\hat{f}_\eta : D := \mathrm{Spf}(\mathcal{O}_K[[q]], (\varpi, q))^{ad}_\eta \to \mathcal{X}^*_{\Gamma_0(p^n)}.$$

The space $D = \mathrm{Spf}(\mathcal{O}_K[[q]], (\varpi, q))^{ad}_\eta$ is the adic open unit disc over K. This is a rigid space, and to fix notation let us recall that its global sections can be written as

$$\mathcal{O}_D(D) = \left\{ \sum_{n\geq 0} a_n q^n \in K[[q]] \text{ such that } |a_n| q^n \to 0 \text{ for all } 0 \leq q < 1 \right\}.$$

The relation of $\mathrm{Spf}(\mathcal{O}_K[[q]], (\varpi, q))$ and D can be described in more classical terms: Namely, the rigid space D is the one associated to $\mathrm{Spf}(\mathcal{O}_K[[q]], (\varpi, q))$ via Conrad's generalisation of Berthelot's rigid generic fibre construction, [3] Theorem 3.1.5 (we need Conrad's generalisation since K might not be discretely valued and §7 of [5] only works for locally noetherian formal schemes).

Proposition 2.3 (Conrad, [3] Theorem 3.2.6)**.** *The morphism $f_\eta : D \to \mathcal{X}^*_{\Gamma_0(p^n)}$ of rigid spaces is an open immersion that identifies D with an open neighbourhood of the cusp.*

Proof. This is the analogue of [3], Theorem 3.2.8 for $\Gamma(N)$ replaced by $\Gamma_1(N)$: Since the cusps are a disjoint union of sections, this is a direct consequence of Theorem 3.2.6. □

Remark 2.4. In fact, [3], Theorem 3.2.6 says much more: It also gives a universal analytic Tate curve over D, and a moduli interpretation of the cusp in terms of generalised elliptic curves.

Lemma 2.5. *Denote by w the map of locally ringed spaces $w : D \to \mathrm{Spec}(\mathbb{Z}(N)[[q]] \otimes \mathcal{O}_K)$ induced by the natural inclusion $\mathbb{Z}(N)[[q]] \otimes \mathcal{O}_K \hookrightarrow \mathcal{O}_D(D)$. Then the following diagram of locally ringed spaces commutes:*

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{Z}(N)[[q]] \otimes \mathcal{O}_K) & \xrightarrow{c} & X^*_{\mathcal{O}_K,\Gamma_0(p^n)} \\ {\scriptstyle w}\uparrow & & \uparrow \\ D & \xrightarrow{c_\eta} & \mathcal{X}^*_{\Gamma_0(p^n)}. \end{array}$$

In particular, the morphism c_η is the one induced by the morphism of locally ringed spaces $D \to \mathrm{Spec}(\mathbb{Z}(N)[[q]] \otimes K) \to X^*_{K,\Gamma_0(p^n)}$ via the universal property of the analytification.

Proof. It suffices to prove this for $\mathcal{O}_K$ replaced by $\mathbb{Z}(N)$, the general case follows by base change. Then we can consider all appearing formal schemes as adic spaces in the sense of Huber. The morphism $f : \mathrm{Spf}(\mathbb{Z}(N)[[q]] \otimes \mathcal{O}_K) \to \mathrm{Spec}(\mathbb{Z}(N)[[q]] \otimes \mathcal{O}_K) \to X^*_{\mathbb{Z}(N),\Gamma_0(p^n)}$ then completes to a morphism $\hat{f}$ of adic spaces, and the universal property of the adification gives a commutative diagram of locally ringed spaces

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{Z}(N)[[q]]) & \xrightarrow{f} & X^*_{\mathbb{Z}(N),\Gamma_0(p^n)} \\ \uparrow & & \uparrow \\ \mathrm{Spf}(\mathbb{Z}_p[\zeta_N][[q]])^{ad} & \xrightarrow{\hat{f}} & \mathfrak{X}^{*ad}_{\mathbb{Z}_p[\zeta_N],\Gamma_0(p^n)} \end{array}$$

The Lemma follows upon taking the fibre over $\mathrm{Spa}(K, \mathcal{O}_K) \to \mathrm{Spec}(\mathcal{O}_K)$. $\square$

We thus have the following moduli interpretation of $\mathring{D}$.

Corollary 2.6. *Let S be an honest adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$ and let $\varphi : S \to X^{an}_{\Gamma_0(p^n)}$ be a morphism corresponding to an elliptic curve E over $\mathcal{O}_S(S)$ with tame and $\Gamma_0(p^n)$-level structure. Then φ factors through the punctured open unit disc $\mathring{D} \to \mathcal{X}_{\Gamma_0(p^n)}$ around the cusp c if and only if E is a Tate curve with level structure corresponding to c, and $q_E \in \mathcal{O}_S(S)$ is topologically nilpotent, that is $v_x(q_E)$ is cofinal in the value group for all $x \in S$.*

Proof. If $\varphi : S \to X^*_{\Gamma_0(p^n)}$ factors through $\mathring{D} \hookrightarrow \mathcal{X}_{\Gamma_0(p^n)}$, then it factors through the map $\mathrm{Spec}(\mathcal{O}_K(q)) \to X^*_{\Gamma_0(p^n)}$ by Lemma 2.5. Consequently, E is a Tate curve and we obtain a parameter $q_E \in \mathcal{O}_S(S)$ as the image of $q \in \mathcal{O}_K((q))$ on global sections. This is topologically nilpotent because $q \in \mathcal{O}_D(D)$ is topologically nilpotent.

Conversely, assume that E is a Tate curve such that $q_E \in \mathcal{O}(S)$ is topologically nilpotent. It suffices to consider the case that $S = \mathrm{Spa}(B, B^+)$ is an affinoid adic space over $\mathrm{Spa}(K, \mathcal{O}_K)$. The condition that q_E is topologically nilpotent then implies that for any x there is n such that $|q_E(x)|^n \leq |\varpi|$. Since S is affinoid and thus quasicompact, we can find n that works for all $x \in S$. Similarly, since E is a Tate curve, the element $q_E \in B$ is a unit and we thus have $0 < |q_E(x)|$ for all $x \in S$. Again by compactness, we can find m such that $|\varpi|^m \leq |q_E|$. But then $q_E^n/\varpi, \varpi^m/q_E \in B^+$ and there is a natural morphism of affinoids

$$(K\langle q, q^n/\varpi, \varpi^m/q\rangle, \mathcal{O}_K\langle q, q^n/\varpi, \varpi^m/q\rangle) \to (B, B^+), \quad q \mapsto q_E$$

through which the map $\mathcal{O}_K((q)) \to B$ defining the Tate curve structure factors. Since the algebra on the left defines an affinoid open of D, this gives the desired factorisation. $\square$

Remark 2.7. Consider the non-archimedean field $(L, \mathcal{O}_L) := (\mathcal{O}_K((q))[1/\varpi], \mathcal{O}_K((q)))$ endowed with the ϖ-adic topology. Then $\mathrm{Spa}(L, \mathcal{O}_L)$ is just a point $\{v_\varpi\}$, and clearly q is not topologically nilpotent. We conclude that the natural morphism

$$\mathrm{Spa}(L, \mathcal{O}_L) \to \mathrm{Spec}\, \mathcal{O}_K((q))$$

does not factor through $\mathring{D}$ even though it corresponds to a Tate curve. Instead, this Tate curve has good reduction and therefore the point lies in the good reduction locus $\mathcal{X} \subseteq X^*$.

2.2 Classification of points of the adic space $\mathcal{X}^*$

Our next goal is to prove a moduli-theoretic decomposition of $\mathcal{X}^*$ and $\mathcal{X}^{an}$. The same classification works for $\mathcal{X}^*_{\Gamma_0(p^n)}$, and in fact for any higher levels, but we only treat the case of $\mathcal{X}^*$ for simplicity.

To motivate the result, recall that we have an open subset $\mathcal{X} \subseteq \mathcal{X}^*$ which parametrises elliptic curves with good reduction, as well as open subsets $\mathring{D} \subseteq X^{an}$ around each cusps. Both of these types of open subsets arises as admissible open subsets of $\mathcal{X}^*$ considered as a rigid space, and in fact they cover the rigid space $\mathcal{X}^*$ set-theoretically: Indeed, it follows from the analogue of [5], Lemma 7.2.5 in the setting of [3] that on the level of the underlying topological spaces, the loci $\mathcal{X}$ and $\mathring{D}$ inside X^* are precisely the preimages of $X_{\mathcal{O}_K/\varpi} \subseteq X^*_{\mathcal{O}_K/\varpi}$ and $Cusps(\Gamma(N)) \subseteq X^*_{\mathcal{O}_K/\varpi}$, respectively, under the specialisation map $\mathrm{sp} : |\mathcal{X}^*| \to |X_{\mathcal{O}_K/\varpi}|$.. This cover, however, is not admissible (cf 7.5.1 of [5]). In terms of adic spaces, this hints at that we are missing rank-2-points of $\mathcal{X}^*$. This is made precise by the following classification of points of $\mathcal{X}^*$.

Theorem 2.8. *Let $x \in \mathcal{X}^*$ be any point, then we are in either of the following cases:*

(a) $x \in \mathcal{X}^*$ is contained in the good reduction locus

(b) $x \in D \hookrightarrow \mathcal{X}^*$ is contained in one of the Tate parameter spaces around the cusps

(c) $x \in \mathcal{X}^* \setminus \mathcal{X}$ is of rank > 1 and its unique height 1 vertical generisation x' is in $\mathcal{X}$.

When we denote by j the global function on X^{an} induced by the morphism $j : X^{an} \to \mathbb{A}^{1,an}$, then the above are respectively equivalent to

(a') $|j(x)| \leq 1$

(b') $|j(x)| > 1$ and its inverse is cofinal in the value group

(c') $|j(x)| > 1$ and its unique rank 1 generisation x' has $|j(x')| = 1$.

Proof. The space $\mathcal{X}^*$ is analytic, hence the valuation v_x is always microbial. This means that x has a unique generisation x' of height 1, so statements (c) and (c') make sense.

The case of the cusps is clear, so let us without loss of generality assume that $x \in \mathcal{X}$.

We start by proving that (a) and (a') are equivalent. Recall that X is the preimage of $\mathbb{A}^1$ under the morphism of $\mathcal{O}_K$-schemes $j : X^* \to \mathbb{P}^1$. Upon formal completion and passing to the adic generic fibre, j becomes j^{an} while $\mathbb{A}^1 \subseteq \mathbb{P}^1$ is sent to the ball $B^1(0) \subseteq \mathbb{A}^{1,an} \subseteq \mathbb{P}^{1,an}$ of valuations with $|j(x)| \leq 1$. Since the adic generic fibre of the completion of $X \subseteq X^*$ is $\mathcal{X}^{gd} \subseteq \mathcal{X}^*$, this shows that $\mathcal{X}^{gd}$ is precisely the preimage of $B^1(0)$ under $j^{an} : X^{an} \to \mathbb{A}^{1,an}$.

Next, let us prove that (b) and (b') are equivalent. We can always find a morphism

$$r_x : \mathrm{Spa}(C, C^+) \to X^{an}$$

where (C, C^+) is a complete algebraically closed non-archimedean field, such that x is in the image of r_x. It thus suffices to show that r_x factors through some $D \hookrightarrow X^*$. By Corollary 2.6 it suffices to show that (b') holds if and only if the elliptic curve E over C that r_x represents is a Tate curve with nilpotent parameter $q_E \in C$.

The image of j in C is precisely the j-invariant j_E of E. Since in a non-archimedean field the elements with cofinal valuation are precisely the topologically nilpotent ones, condition (b') is equivalent to $j_E \neq 0$ and j_E^{-1} being topologically nilpotent. We can now argue like in the classical case of p-adic fields to see that this is equivalent to E being a Tate curve with q_E topologically nilpotent: If E is a Tate curve with q_E topologically nilpotent, then $j_E = 1/q_E + 744 + \cdots \neq 0$ has valuation $|j_E| = |1/q_E|$ in C and thus j_E satisfies (b'). To see the converse, recall that in the formal Laurent series ring $\mathbb{Z}((q))$ the formula $j(q) = 1/q + 744 + 196884q^2 + \dots$ reverses to

$$q(j^{-1}) = j^{-1} + 744j^{-2} + 750420j^{-3} + \cdots \in \mathbb{Z}[[j^{-1}]].$$

If now j_E^{-1} is topologically nilpotent, the above series converges in C and we obtain a topologically nilpotent element $q_E \in C^\times$ with $j_E = 1/q_E + 744 + \cdots = j(q_E)$. The Tate curve E_{q_E} over C thus has the same j-invariant as E, and since C is algebraically closed we conclude that $E \cong Tate(q_E)$. Thus E is a Tate curve with topologically nilpotent parameter $q_E \in C$, as desired. This shows that (b) and (b') are equivalent.

Next let us show that (c') holds if and only if (a') and (b') don't hold. Recall that we always have a unique height 1 vertical generisation x'. Clearly $|j(x)| \neq 0$ if and only if $|j(x')| \neq 0$, and if in this case $|j(x)|^{-1}$ is cofinal then $|j(x')|^{-1}$ is cofinal. This implies that (b') and (c') can't hold at the same time. On the other hand, if $|j(x)| > 1$, then either $|j(x')| = 1$, or $|j(x')| > 1$ in which case $|j(x')|^{-1} < 1$ is cofinal because $v_{x'}$ has height 1. This shows that if $|j(x)| > 1$ then we are either in case (b') or in (c').

It remains to prove that (c) and (c') are equivalent. By the equivalence of (a) and (a') we know that $x \notin \mathcal{X}$ is equivalent to $|j(x)| > 1$, and that $x' \in \mathcal{X}$ is equivalent to $|j(x')| \leq 1$. Since $|j(x)| > 1$, the generisation satisfies $|j(x')| \geq 1$, and therefore $|j(x')| \leq 1$ implies $|j(x')| = 1$. This finishes the proof of the Theorem. □

Example 2.9. Let us work out an example for an elliptic curve corresponding to a point of type (c): Let $R_{\geq 0} \times \gamma^{\mathbb{Z}}$ be the totally ordered group where γ is such that $x < \gamma < 1$ for all $x \in \mathbb{R}_{<1}$. Consider the field $L = \mathcal{O}_K((q))[1/\varpi]$ equipped with the valuation

$$x_{1^-} : \mathcal{O}_K((q))[1/\varpi] \to \mathbb{R}^{\geq 0} \times \gamma^{\mathbb{Z}}, \quad \sum a_n q^n \mapsto \max_{n \in \mathbb{Z}} |a_n| \gamma^n.$$

Denote by m_K the maximal ideal of $\mathcal{O}_K$, then the valuation ring of x_{1^-} is

$$\mathcal{O}_L^+ = \left\{ \sum_{n \gg -\infty}^{\infty} a_n q^n \middle| a_n \in \mathcal{O}_K \text{ for all } n \geq 0 \text{ and } a_n \in m_K \text{ for all } n < 0 \right\}.$$

Indeed, we have $v_{1^-}(\sum_{n \gg -\infty}^{\infty} a_n q^n) \leq 1$ if and only if $|a_n|\gamma^n \leq 1$ for all n. For $n \geq 0$ we have $|a_n|\gamma^n \leq 1$ if and only if $|a_n| \leq 1$, that is $a_n \in \mathcal{O}_K$. For $n < 0$, on the other hand, γ^n is "infinitesimally" bigger than 1, so that $|a_n|\gamma^n \leq 1$ if and only if $|a_n| < 1$, that is $a_n \in m_K$.

The Tate curve $Tate(q)$ over L with any of its $\Gamma(N)$-structures gives rise to a map

$$t_{1^-} : \mathrm{Spa}(L, \mathcal{O}_L^+) \to \mathcal{X}^*$$

which we claim lands neither in $\mathcal{X}$ nor in any of the Tate parameter spaces $D \subseteq \mathcal{X}^*$. Indeed, the j-invariant of $Tate(q)$ is

$$j = 1/q + 744 + \cdots \notin \mathcal{O}_L^+ \qquad (1)$$

which is not contained in $\mathcal{O}_L^+$ by the above description. This shows that $Tate(q)$ does not extend to an elliptic curve over $\mathcal{O}_L^+$. On the other hand, q is not nilpotent in L and so the map t does not factor through any of the open immersions $D \hookrightarrow X^*$.

Theorem 2.8 explains this as follows: We have

$$\mathrm{Spa}(L, \mathcal{O}_L^+) = \{x = x_{1^-}, x'\}$$

where x' is the unique height 1 vertical generisation of x given by

$$x' : \mathcal{O}_K((q))[1/\varpi] \to \mathbb{R}^{\geq 0}, \quad \sum a_n q^n \mapsto \max_{n \in \mathbb{Z}} |a_n|$$

with valuation ring $\mathcal{O}_L = \mathcal{O}'_K((q))$ containing $\mathcal{O}_L^+$. We now see from equation 1 that

$$|j(x)| = \gamma^{-1} < 1 \text{ while } |j(x')| = 1.$$

This shows that t^{-1} sends x to one of the points of type (c) in Theorem 2.8, while its generisation x' goes to the point of $\mathcal{X}$ defined by Remark 2.7.

2.3 Tate parameter spaces in the anticanonical tower

Next, we want to look at the behaviour of the Tate parameter spaces in the anticanonical tower. For this we first recall the situation at the cusps on the level of schemes:

Consider the morphism $f : X^*_{\Gamma_0(p)} \to X^*$. Over each cusp of X^* there are precisely two cusps of $X^*_{\Gamma_0(p)}$: One is called the étale cusp, it corresponds to the $\Gamma_0(p)$-level structure $\mu_p \subseteq \mathrm{T}[p]$ on the Tate curve. The other is the ramified cusp, it corresponds to the level structure $\langle q^{1/p} \rangle \subseteq \mathrm{T}[p]$. The names reflect that $X^*_{\Gamma_0(p)} \to X^*$ is étale at the one sort of cusps, but is ramified at the other. More precisely, over the étale cusp the morphism induced on completions is given by

$$\mathcal{O}_K[[q]] \to \mathcal{O}_K[[q]], q \mapsto q$$

whereas over the ramified cusp it is

$$\mathcal{O}_K[[q]] \to \mathcal{O}_K[[q]], q \mapsto q^p$$

For higher level structures $\Gamma_0(p^n)$, the curve $X^*_{\Gamma_0(p^n)} \to X^*$ has more cusps of different degrees of ramification $d = p^i$ with $i = 0, \dots, n$, and corresponding morphisms on completions given by $q \mapsto q^d$. There is, however, exactly one étale cusp, corresponding to the level structure μ_{p^n}, and exactly one purely ramified one, corresponding to $\langle q^{1/p^n} \rangle$. Relatively over the morphism $X^*_{\Gamma_0(p^n)} \to X^*_{\Gamma_0(p)}$, all the cusps upstairs lie over the étale cusps of $X^*_{\Gamma_0(p)}$, except for the purely ramified one, which lies over the ramified cusp of $X^*_{\Gamma_0(p)}$.

We note the following consequence for the Tate parameter spaces:

Proposition 2.10.

1. *The cusps of $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ are precisely the purely ramified cusps of $\mathcal{X}^*_{\Gamma_0(p^n)}$. Let c be any such cusp. Then for any honest adic space S over $(K, \mathcal{O}_K)$, the S-points of $f(c) : \mathring{D} \hookrightarrow \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a$ correspond functorially to Tate curves over $\mathcal{O}(S)$ with topologically nilpotent parameter $q \in \mathcal{O}(S)$, a $\Gamma(N)$-structure corresponding to c_0 and a choice of p^n-th root of q defining a subgroup $\langle q^{1/p^n} \rangle \subseteq Tate(q)$.*

2. *The forgetful map $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon) \to \mathcal{X}^*_{\Gamma_0(p^{n-1})}(\epsilon)$ gives a bijection of the cusps of both spaces. For any cusp of $\mathcal{X}^*_{\Gamma_0(p^{n-1})}(\epsilon)$ and its corresponding cusp of $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)$, the Tate parameter spaces associated to these cusps fit into Cartesian diagrams*

$$\begin{array}{ccc} D & \xrightarrow{q \mapsto q^p} & D \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^{n-1})}(\epsilon)_a. \end{array}$$

Proof. Since the canonical subgroup of the Tate curve is given by $\mu_p \subseteq \mathrm{T}[p]$, the cusps contained in the anticanonical locus are precisely the ramified ones. But the cusps of $X^*_{\Gamma_0(p^n)}$ over the ramified cusps of $X^*_{\Gamma_0(p)}$ are precisely the purely ramified ones. This proves 1.

The diagrams in 2 commutes because by construction it is the generic fibre of a commutative diagram of formal schemes. Since the morphisms are open immersions, it suffices to check that it is Cartesian on the level of points. But this follows from Lemma 2.6. □

2.4 Tate parameter spaces of $\mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$

The aim of this section is to describe Tate parameter spaces $D \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^n)}$ like in Theorem 2.3. Since the integral theory of cusps for $\Gamma_1(p^n)$ is complicated (see §4.2 of [4] for a thorough discussion), Theorem 4.16 of [3] does not apply immediately. Instead we shall use ad hoc methods to deduce the desired description from the case of $\Gamma_0(p^n)$. We shall restrict attention to the cusps of $\mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$.

Proposition 2.11. *Let c be a cusp of $\mathcal{X}^*$ and denote by $f : D \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ the corresponding Tate parameter space.*

1. *There is a Cartesian diagram of $(\mathbb{Z}/p^n\mathbb{Z})^\times$-equivariant maps*

$$\begin{array}{ccc} \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D & \longrightarrow & D \\ \big\downarrow & & \big\downarrow f \\ \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a. \end{array}$$

2. *The projection $\pi : (\mathbb{Z}/p^{n+1}\mathbb{Z})^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times$ induces a Cartesian diagram*

$$\begin{array}{ccc} \underline{(\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \times D & \longrightarrow & \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^*_{\Gamma_1(p^{n+1})}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a \end{array}$$

where the morphism on top is given by $(a, q) \mapsto (\pi(a), q^p)$.

Proof. We first construct a section $D \to X^*_{\Gamma_1(p^n)}$: The purely ramified cusp corresponds to the choice of $\langle q^{1/p}\rangle \subseteq \mathrm{T}(q^{p^n N})$ as a $\Gamma_0(p^n)$-structure. This can be lifted canonically to the $\Gamma_1(p^n)$-structure given by the generator $q^{1/p}$ of $\langle q^{1/p}\rangle$, defining a canonical lift

$$\begin{array}{ccc} & & \mathrm{Spec}(\mathbb{Z}[1/N, \zeta_N][[q]] \otimes K) \\ & \swarrow & \big\downarrow \\ X^*_{\Gamma_1(p^n)} & \longrightarrow & X^*_{\Gamma_0(p^n)}. \end{array}$$

The natural morphism $D \to \mathrm{Spec}(\mathbb{Z}[1/N, \zeta_N][[q]] \otimes K) \to X^*_{\Gamma_0(p^n)}$ from Corollary 2.5 together with the universal property of the analytification now give rise to a section

$$\begin{array}{ccc} & & D \\ & \overset{f}{\swarrow} & \big\downarrow \\ \mathcal{X}^*_{\Gamma_1(p^n)} & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}. \end{array}$$

Since $\mathcal{X}^*_{\Gamma_1(p^n)} \to \mathcal{X}^*_{\Gamma_0(p^n)}$ is a Galois torsor with group $(\mathbb{Z}/p^n\mathbb{Z})^\times$, this already implies that the natural morphism $(\mathbb{Z}/p^n\mathbb{Z})^\times \times D \to X^*_{\Gamma_1(p^n)} \times_{X^*_{\Gamma_0(p^n)}} D$ is an isomorphism.

The second part follows from the fact that the morphism $\mathcal{X}^*_{\Gamma_1(p^n)} \to \mathcal{X}^*_{\Gamma_0(p^n)}$ is equivariant with respect to the morphism of Galois groups $(\mathbb{Z}/p^{n+1}\mathbb{Z})^\times \to (\mathbb{Z}/p^n\mathbb{Z})^\times$. □

2.5 Tate parameter spaces of $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$

Next, we look at what happens with the cusps in the transition $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$.

Let us fix notation for the *left* action of $\Gamma_1(p^n, \mathbb{Z}/p^n\mathbb{Z})$ on $\mathcal{X}^*_{\Gamma(p^n)}$ in terms of moduli: For any $\gamma \in \Gamma_1(p^n, \mathbb{Z}/p^n\mathbb{Z})$ it is given by sending a trivialisation $(\mathbb{Z}/p^n\mathbb{Z})^2 \xrightarrow{\sim} E[p^n]$ to

$$(\mathbb{Z}/p^n\mathbb{Z})^2 \xrightarrow{\gamma^\vee} (\mathbb{Z}/p^n\mathbb{Z})^2 \xrightarrow{\sim} E[p^n]$$

where $\gamma^\vee = \det(g)\gamma^{-1}$. Here the inverse is necessary to indeed obtain a left action, and the additional twist by $\det(g)$ is necessary to ensure that action on the fibres of the map $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$ is given by matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ rather than $\left(\begin{smallmatrix} 1 & * \\ 0 & * \end{smallmatrix}\right)$.

Definition 2.12. For any $0 \le m \le n \in \mathbb{N}$, we denote by $\Gamma_0(p^m, \mathbb{Z}/p^n\mathbb{Z}) \subseteq \mathrm{GL}_2(p^m, \mathbb{Z}/p^n\mathbb{Z})$ the subgroup of matrices which are of the form $\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right)$ with $c \equiv 0 \bmod p^m$. We similarly define $\Gamma_0(p^m, \mathbb{Z}_p)$.

The forgetful map $X^*_{\Gamma(p^n)} \to X^*_{\Gamma_0(p)}$ is given by reducing $(\mathbb{Z}/p^n\mathbb{Z})^2 \xrightarrow{\sim} E[p^n] \bmod p$ to $(\mathbb{Z}/p\mathbb{Z})^2 \xrightarrow{\sim} E[p]$ and sending it to the subgroup generated by $(1,0)$. Consequently, the action of $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$ leaves the forgetful morphism $X^*_{\Gamma(p^n)} \to X^*_{\Gamma_0(p)}$ invariant. We see from this that the action of $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$ fixes the forgetful map to $\mathcal{X}^*_{\Gamma_0(p)}$, and thus restricts to an action on $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$.

Definition 2.13. Denote by $\Gamma_1(p^n, \mathbb{Z}/p^n\mathbb{Z}) \subseteq \Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z}) \subseteq \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ the subgroup of matrices which are of the form $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$. These are precisely the matrices for which the action on $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ commutes with the forgetful map to $\mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a$.

Proposition 2.14. *Let c be a cusp of $\mathcal{X}^*$ and denote by $f : D \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ the corresponding Tate parameter space.*

1. *Depending on our chosen primitive root ζ_{p^n}, there is a canonical Cartesian diagram*

$$\begin{array}{ccc} \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a. \end{array}$$

   where the map on the left is $\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$-equivariant for the trivial action on D.

2. *Let c be a cusp of $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ and let c_γ be the cusp of $\mathcal{X}^*_{\Gamma(p^n)}(0)_a$ over c determined by $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$. Then for any honest adic space S over $(K, \mathcal{O}_K)$, the S-points of the map $f(c_\gamma) : \mathring{D} \hookrightarrow \mathcal{X}_{\Gamma(p^n)}(\epsilon)_a$ correspond functorially to Tate curves with topologically nilpotent parameter $q \in \mathcal{O}(S)$, a $\Gamma_1(N)$-structure determined by c, and the basis $(q^{d/p^n}, q^{-b/p^n}\zeta_{p^n}^a)$ of $E[p^n]$, where q^{1/p^n} is the p^n-th root of q determined by c.*

3. *The morphisms $\pi : \Gamma_0(p^{n+1}, \mathbb{Z}/p^{n+1}\mathbb{Z}) \to \Gamma_0(p^{n+1}, \mathbb{Z}/p^{n+1}\mathbb{Z})$ induced by the reduction maps $\mathrm{GL}_2(\mathbb{Z}/p^{n+1}\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ give rise to a Cartesian diagram*

$$\begin{array}{ccc} \underline{\Gamma_0(p^{n+1}, \mathbb{Z}/p^{n+1}\mathbb{Z})} \times D & \longrightarrow & \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^*_{\Gamma(p^{n+1})}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \end{array}$$

where the morphism on top is given by $(\gamma, q) \mapsto (\pi(\gamma), q^p)$.

Proof. 1. Recall that over the purely ramified cusp there is a canonical splitting

$$\begin{array}{ccc} & & \mathrm{Spec}(\mathbb{Z}[1/N, \zeta_{p^n}, \zeta_N][[q]] \otimes K) \\ & \swarrow & \downarrow \\ X^*_{\Gamma(p^n)} & \longrightarrow & X^*_{\Gamma_1(p^n)}. \end{array}$$

constructed as follows: As we have already seen, there is a canonical $\Gamma_1(p^n)$-structure given by the generator q of $\langle q \rangle \subseteq T[p^n]$. If T is the Tate curve $T = \mathrm{T}(q^{p^n N})$ over $\mathbb{Z}[1/N, \zeta_N]((q))$, the Weil pairing gives a pairing $e_{p^n} : T[p^n] \times T[p^n] \to \mu_{p^n}$. Specialising the Weil pairing at $q \in T[p^n]$, we obtain an isomorphism $e(q, -) : C_n \to \mu_{p^n}$ where C_n is the canonical subgroup of T. The preimage of ζ_{p^n} under $e(q, -)$ gives the desired second basis vector of $T[p^n]$. This defines a canonical $\Gamma(p^n)$-structure on $T[p^n]$, which via normalisation gives rise to the morphism in diagram (1).

The universal property of the analytification then gives rise to a map $g : D \to \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ which gives the Cartesian diagram in 1 using that $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ is a Galois torsor with group $\underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})}$.

2. The cusp label $1 \in \Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$ corresponds to the basis $(q^{1/p^n}, \zeta_{p^n})$. The action by $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ sends this to

$$\left(\begin{smallmatrix} d & 0 \\ -b & a \end{smallmatrix}\right) \begin{pmatrix} q^{1/p^n} \\ \zeta_{p^n} \end{pmatrix} = \begin{pmatrix} q^{d/p^n} \\ q^{-b/p^n} \zeta_{p^n}^a \end{pmatrix}$$

where $\left(\begin{smallmatrix} d & 0 \\ -b & a \end{smallmatrix}\right) = \gamma^\vee$. This gives the desired basis.

3. This follows from Lemma 2.10 since $X^*_{\Gamma(p^n)} \to X^*_{\Gamma(p^{n-1})}$ is equivariant via π. ◻

We can now fit all descriptions of Tate parameter spaces together for a complete description of Tate parameter spaces at finite level.

Corollary 2.15. *Let c be any cusp of $\mathcal{X}^*$ and let $D \hookrightarrow \mathcal{X}^*$ be its Tate parameter space.*

1. *Depending on our choice of $\zeta_{p^n} \in \mathcal{O}_K$, there is a canonical diagram*

$$\begin{array}{ccccccc} \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D & \longrightarrow & \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D & \longrightarrow & D & \xrightarrow{q \mapsto q^{p^n}} & D \\ \downarrow{\scriptstyle \varphi} & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*(\epsilon) \end{array}$$

of Cartesian squares, where the map on the top left is induced by the map

$$\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z}) \to (\mathbb{Z}/p^n\mathbb{Z})^\times, \quad \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto d.$$

Proof. The Cartesian diagram on the left exists as a consequence of 2.14 together with the fact that $X^*_{\Gamma(p^n)} \to X^*_{\Gamma_1(p^n)}$ is equivariant with respect to the map $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto d$. The square in the middle is Proposition 2.11, the square on the right is Corollary 2.10.(3). ◻

Definition 2.16. In the situation of the Proposition, let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$. Then specialising $\varphi : \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D \hookrightarrow \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ at the point of $\underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})}$ defined by γ defines a morphism that we shall denote by $\varphi(\gamma) : D \hookrightarrow \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$. By equivariance, we have for $\gamma' \in \Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$

$$\varphi(\gamma'\gamma) = \gamma'\varphi(\gamma). \tag{2}$$

2.6 The action of $\Gamma_0(p)$ on the cusps of $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$

While Proposition 2.14 describes the $\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$-action on the Tate parameter spaces of $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$, this space has an action of the larger group $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$. While the action of $\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$ just permutes the different copies of D at different cusps, the action of $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$ has a non-trivial effect on the Tate parameter space, because it also takes into account isomorphisms of Tate curves induced by sending $q \mapsto q\zeta_{p^n}$, as we shall now discuss.

Proposition 2.17. *Over any cusp c of $\mathcal{X}^*$, the $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$-action on $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ restricts to an action on $\varphi : \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D \hookrightarrow \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ where it can be described as follows: Equip $\underline{\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})} \times D$ with a right action by $p\mathbb{Z}/p^n\mathbb{Z}$ via $(\gamma, q) \mapsto (\gamma \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right), \zeta_{p^n}^{h/N} q)$, then*

$$(\underline{\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})} \times D)/(p\mathbb{Z}/p^n\mathbb{Z}) = \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D$$

and the left action of $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$ is the natural left action induced on the quotient.

Explicitly, for any $\gamma_1 \in \Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$, the action is given by

$$\begin{aligned} \gamma_1 : \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D &\xrightarrow{\sim} \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D \\ \gamma_2, q &\mapsto \begin{pmatrix} \det(\gamma_3)/d_3 & b_3 \\ 0 & d_3 \end{pmatrix}, \zeta_{p^n}^{-\frac{c_3}{d_3 N}} q \end{aligned}$$

where $\gamma_3 = \left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right) := \gamma_1 \cdot \gamma_2$.

Proof. Recall that the reason for the pull-back of $D \hookrightarrow \mathcal{X}^*_{\Gamma_0(p)}$ to $\mathcal{X}^*_{\Gamma(p^n)}$ being of the form $\underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D$ even though $\mathcal{X}_{\Gamma(p^n)} \to \mathcal{X}_{\Gamma_0(p)}$ has larger Galois group $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$ is that in the step from $\mathcal{X}^*$ to $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ the isomorphism $D \to D$, $q \mapsto \zeta_{p^n}^h q$ for any $h \in \mathbb{Z}/p^n\mathbb{Z}$ induces an automorphism of the Tate curves $\mathrm{T}(q^{p^n N})$ that sends the anti-canonical $\Gamma_0(p^n)$ level structure $\langle q^N \rangle$ to $\langle \zeta_{p^n}^{hN} q^N \rangle$. For the action of $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$, this means the following:

Consider the Tate parameter $\varphi(\mathrm{id}) : D \hookrightarrow \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ corresponding to $\mathrm{id} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, that is to the isomorphism $\alpha : (\mathbb{Z}/p^n\mathbb{Z})^2 \to Tate(q^{p^n N})[p^n]$ that sends $(1,0) \mapsto q^N$ and $(0,1) \mapsto \zeta_{p^n}$. Then the action of $\gamma_1 = \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right)$ sends this to the isomorphism $\alpha \circ \gamma^\vee$ defined by $(1,0) \mapsto \zeta_{p^n}^{-h} q^N$ and $(0,1) \mapsto \zeta_{p^n}$. The isomorphism $D \to D$, $q \mapsto \zeta_{p^n}^{-h/N} q$ identifies this with the basis (q, ζ_{p^n}). We see from this that the following diagram commutes:

$$\begin{array}{ccccc} \langle q^N \rangle & \longrightarrow & \mathring{D} & \xhookrightarrow{\varphi(\mathrm{id})} & \mathcal{X}_{\Gamma(p^n)}(\epsilon)_a \\ \downarrow & & \downarrow {\scriptstyle q \mapsto \zeta_{p^n}^{-h/N} q} & & \downarrow {\scriptstyle \gamma_1} \\ \langle \zeta_{p^n}^{-h} q^N \rangle & \longrightarrow & \mathring{D} & \xhookrightarrow{\varphi(\mathrm{id})} & \mathcal{X}_{\Gamma(p^n)}(\epsilon)_a. \end{array}$$

We have thus computed the action of $\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)$ on the component of $\underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D$ defined by $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, namely we have $\gamma_1 \varphi(\mathrm{id}) = \varphi(\mathrm{id}) \circ (q \mapsto \zeta_{p^n}^{-h/N} q^N)$

In the general case, one can decompose any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$ as

$$\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \begin{pmatrix} \det(\gamma)/d & b \\ 0 & d \end{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ c/d & 1 \end{smallmatrix}\right). \tag{3}$$

Combined with the equivariance of φ under $\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$, we can compute from this the action of $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$: Let $\gamma_1, \gamma_2, \gamma_3$ be like in the statement of the Proposition, then

$$\begin{aligned} \gamma_1 \varphi(\gamma_2) &= \gamma_1 \gamma_2 \varphi(\mathrm{id}) = \gamma_3 \varphi(\mathrm{id}) = \begin{pmatrix} \det(\gamma_3)/d_3 & b_3 \\ 0 & d_3 \end{pmatrix} \left(\begin{smallmatrix} 1 & 0 \\ c_3/d_3 & 1 \end{smallmatrix}\right) \varphi(\mathrm{id}) \\ &= \begin{pmatrix} \det(\gamma_3)/d_3 & b_3 \\ 0 & d_3 \end{pmatrix} \varphi(\mathrm{id}) \circ (q \mapsto \zeta_{p^n}^{-c_3/d_3 N} q^N) \\ &= \varphi(\begin{pmatrix} \det(\gamma_3)/d_3 & b_3 \\ 0 & d_3 \end{pmatrix}) \circ (q \mapsto \zeta_{p^n}^{-c_3/d_3 N} q^N). \end{aligned}$$

This gives the desired explicit formula. ☐

3 Tate parameter spaces for infinite level modular curves

We now pass to infinite level, starting with level $\Gamma_0(p^\infty)_a$.

Lemma 3.1. *The cusps of $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ are a disjoint union of points which correspond one-to-one to the cusps of $\mathcal{X}^*$ under the map $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*$.*

Proof. The forgetful maps $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a \to \mathcal{X}^*(\epsilon)$ induce a one-to-one correspondence on cusps at every very level p^n by Proposition 2.10.3. The Lemma then follows from the identification of topological spaces $|\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a| = \varprojlim |\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a|$. ☐

Corollary 3.2. *Let (R, R^+) be a perfectoid $(K, \mathcal{O}_K)$-algebra. The set $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a(R, R^+)$ is in functorial bijection with isomorphism classes of triples $(E, \alpha_N, (D_n)_{n\in\mathbb{N}})$ of an elliptic curve E over R that is ϵ-nearly ordinary, together with a Γ^p-structure α_N and a collection of anticanonical cyclic subgroups $D_n \subseteq E[p^n]$ of rank p^n for all n that are compatible in the sense that $D_n = D_{n+1}[p^n]$. Equivalently, one could view $(D_n)_{n\in\mathbb{N}}$ as a p-divisible subgroup of $E[p^\infty]$ of height 1 such that D_1 is anticanonical.*

Proof. Since (R, R°) is perfectoid, one has $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a(R, R^\circ) = \varprojlim \mathcal{X}_{\Gamma_0(p^n)}(\epsilon)_a(R, R^\circ)$ by [12], Proposition 2.4.5. The statement thus follows from Lemma 2.1. ☐

Definition 3.3. For the sake of brevity, we shall call the data of the collection of anticanonical D_n an **anticanonical $\Gamma_0(p^\infty)$-structure** or just a **$\Gamma_0(p^\infty)_a$-structure**. We will also call the p-divisible group $D = (D_n)_{n\in\mathbb{N}}$ a $\Gamma_0(p^\infty)_a$-structure.

Recall that in the very beginning, we have chosen a flexible tame level Γ^p. We now want to briefly look at what happens if we make change this level: More precisely, let Γ'^p be any other tame level structure with conditions like in §2.1 and assume that Γ'^p and Γ^p are related via a morphism

$$f : X_{\mathbb{Z}_p,\Gamma'^p} \to X_{\mathbb{Z}_p,\Gamma^p}$$

of affine flat $\mathbb{Z}_p$ schemes that extends to the cusps. The constructions we have made so far also apply to the base change $X_{\Gamma'^p}$ to $\mathcal{O}_K$ and we thus obtain another modular curve at infinite level $\mathcal{X}^*_{\Gamma'^p\Gamma_0(p^\infty)}(\epsilon)_a$. The following Lemma is related to Theorem III.3.18 of [11]:

Proposition 3.4. *Assume we are in either of the following situations:*

(a) we have $\Gamma'^p \subseteq \Gamma^p$, and f is the forgetful morphism $f : X_{\Gamma'^p} \to X_{\Gamma^p}$,

(b) we have $\Gamma'^p = \Gamma^p = \Gamma_1(N)$ and f is the action of some $d \in \mathbb{Z}/N\mathbb{Z}$ on $X_{\Gamma_1(N)}$.

(c) we have $\Gamma'^p = \Gamma^p \times \Gamma_0(M)$ for some M coprime to N and p, and f is the morphism

$$f : X_{\Gamma^p\times\Gamma_0(M)} \to X_{\Gamma^p}, \quad (E, \alpha_N, G) \mapsto (E/G, \alpha_N/G).$$

Let $0 \le \epsilon < 1/2$ and consider the analytification f^{an} of f. Then

1. *The map f^{an} restricts to $f^{an} : \mathcal{X}^*_{\Gamma'^p}(\epsilon) \to \mathcal{X}^*_{\Gamma^p}(\epsilon)$.*
2. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{X}^*_{\Gamma'^p}(p^{-1}\epsilon) & \xrightarrow{f^{an}} & \mathcal{X}^*_{\Gamma^p}(p^{-1}\epsilon) \\ \downarrow \phi & & \downarrow \phi \\ \mathcal{X}^*_{\Gamma'^p}(\epsilon) & \xrightarrow{f^{an}} & \mathcal{X}^*_{\Gamma^p}(\epsilon). \end{array}$$

3. *In the limit, this induces a map of perfectoid spaces* $f_\infty : \mathcal{X}^*_{\Gamma'^p\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma^p\Gamma_0(p^\infty)}(\epsilon)_a$.
4. *The map* f_∞ *has a canonical formal model* $\mathfrak{f}_\infty : \mathfrak{X}^*_{\Gamma'^p\Gamma_0(p^\infty)}(\epsilon)_a \to \mathfrak{X}^*_{\Gamma^p\Gamma_0(p^\infty)}(\epsilon)_a$.

Proof. To see the first part, we note that the Hasse invariant can be controlled: For (a) and (b) the Hasse invariant is unchanged, and in the case of (c), for any E with good reduction, a lift for the Hasse invariant for E/G is given by the image of any lift of the Hasse invariant of E under the morphism $\omega_E^{\otimes(p-1)} \to \omega_{E/G}^{\otimes(p-1)}$ induced by the dual isogeny $E/G \to E$. This also shows that one can construct a canonical formal model.

$$\mathfrak{f} : \mathfrak{X}^*_{\Gamma'^p}(\epsilon) \to \mathfrak{X}^*_{\Gamma^p}(\epsilon). \tag{4}$$

Next, one checks that the following diagram of flat formal schemes commutes

$$\begin{array}{ccc} \mathfrak{X}^*_{\Gamma'^p}(p^{-1}\epsilon) & \xrightarrow{\mathfrak{f}} & \mathfrak{X}^*_{\Gamma^p}(p^{-1}\epsilon) \\ \downarrow \phi & & \downarrow \phi \\ \mathfrak{X}^*_{\Gamma'^p}(\epsilon) & \xrightarrow{\mathfrak{f}} & \mathfrak{X}^*_{\Gamma^p}(\epsilon) \end{array}$$

of flat formal schemes commutes. This can be checked on the generic fibre, which amounts to checking 2: Away from the cusps this can be seen from the moduli interpretation (where for (c) we use that the canonical subgroup of E/G is the image of $C_1 \subseteq E \to E/G$). Over the cusps it then follows from an explicit consideration of Tate parameter spaces.

In the inverse limit over ϕ, we obtain the map

$$\mathfrak{f}_\infty : \mathfrak{X}^*_{\Gamma'^p\Gamma_0(p^\infty)}(\epsilon)_a \to \mathfrak{X}^*_{\Gamma^p\Gamma_0(p^\infty)}(\epsilon)_a$$

and obtain the morphism f_∞ from 3 as the generic fibre. □

3.1 The perfectoid Tate parameter space at level $\Gamma_0(p^\infty)$

Next, we have a closer look at the cusps in the anticanonical tower and at infinite level. As usual when working with Tate curves, we assume for simplicity that $\Gamma^p = \Gamma(N)$.

Lemma 2.10 shows that over any cusp of $\mathcal{X}^*(\epsilon)$ there is a tower of Cartesian squares

$$\begin{array}{ccccc} \dots & \xrightarrow{q\mapsto q^p} & D & \xrightarrow{q\mapsto q^p} & D \\ & & \downarrow & & \downarrow \\ \dots & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*(\epsilon). \end{array} \tag{5}$$

We first look at the limit of the tower in the upper line.

Proposition 3.5. *Consider the tower of analytic adic spaces* $\cdots \xrightarrow{q\mapsto q^p} D \xrightarrow{q\mapsto q^p} D$.

1. *There is a unique perfectoid space* D_∞ *such that* $D_\infty \sim \varprojlim_{q\mapsto q^p} D$.
2. *The space* D_∞ *can be described as the open perfectoid unit disc, that is the subspace of the perfectoid unit disc* $\mathrm{Spa}(K\langle q^{1/p^\infty}\rangle, \mathcal{O}_K\langle q^{1/p^\infty}\rangle)$ *defined by the condition* $|q| < 1$.

3. *The global sections of D are*

$$\mathcal{O}_D(D) = \left\{ \sum_{n\in\mathbb{Z}[1/p]_{\geq 0}} a_n q^n \in K[[q]] \;\middle|\; \begin{array}{c} |a_n|q^n \to 0 \text{ for all } 0 \leq q < 1, \\ |a_n| \to 0 \text{ on bounded intervals} \end{array} \right\}$$

where the second condition means that for any $\delta > 0$ and for any bounded interval $I \subseteq \mathbb{Z}[1/p]_{\geq 0}$ there are only finitely many n such that $|a_n| > \delta$.

4. *Denote by $\mathcal{O}_K[[q^{1/p^\infty}]]$ the (ϖ, q)-adic completion of $\varinjlim_{n\in\mathbb{N}} \mathcal{O}_K[[q^{1/p^n}]]$. Then the space D_∞ is the adic generic fibre of the formal scheme $\mathrm{Spf}(\mathcal{O}_K[[q^{1/p^\infty}]], (\varpi, q))$.*

Proof. This is easy to see using the closed perfectoid unit disc, but we instead choose to work with an explicit affinoid perfectoid cover of D_∞ that we need later: The space D can be covered by $D(|q|^{p^n} \leq |\varpi|) = \mathrm{Spa}(K\langle q/\varpi^{1/p^n}\rangle, \mathcal{O}_K\langle q/\varpi^{1/p^n}\rangle)$ for $n \to \infty$. Under the morphism $D \to D$, $q \mapsto q^p$ these pull back to $D(|q|^{p^{n+1}} \leq |\varpi)$, the corresponding morphism on algebras being

$$\mathcal{O}_K\langle q/\varpi^{1/p^n}\rangle \to \mathcal{O}_K\langle q/\varpi^{1/p^{n+1}}\rangle, \quad q \mapsto q^p$$

Here by $\mathcal{O}_K\langle q/\varpi^{1/p^n}\rangle = \mathcal{O}_K\langle q, q/\varpi^{1/p^n}\rangle$ we denote the algebra $\mathcal{O}_K\langle q, Y\rangle/(q - Y\varpi^{1/p^n})$ of function which converge on the closed disc of radius $|\varpi^{1/p^n}|$. This algebra is isomorphic over $\mathcal{O}_K$ to $\mathcal{O}_K\langle Y\rangle$. When we take the direct limit of these spaces, and complete p-adically, we obtain an algebra that we denote by

$$\mathcal{O}_K\langle (q/\varpi^{1/p^n})^{1/p^\infty}\rangle = \left(\varinjlim_{m\in\mathbb{N}, q\mapsto q^p} \mathcal{O}_K\langle (q/\varpi^{1/p^n})^{1/p^m}\rangle \right)^\wedge$$

which is isomorphic to $\mathcal{O}_K\langle Y^{1/p^\infty}\rangle$. From this description it is easy to check that the algebra $K\langle q^{1/p^\infty}/\varpi^{1/np^\infty}\rangle$ that we get from inverting p is perfectoid. It is then clear that the perfectoid space $D_\infty(q/\varpi^{1/p^n}) := \mathrm{Spa}(K\langle (q/\varpi^{1/p^n})^{1/p^\infty}\rangle, \mathcal{O}_K\langle (q/\varpi^{1/p^n})^{1/p^\infty}\rangle)$ is the tilde-limit

$$D_\infty(q/\varpi^{1/p^n}) \sim \varprojlim_{q\mapsto q^p} D(|q|^{p^n} \leq \varpi^{1/np^m}). \tag{6}$$

Increasing n, it is immediate from the universal property of the perfectoid tilde-limit that the U_n glue together to give the desired perfectoid space D_∞.

Showing 4 isn't quite formal because tilde-limits don't necessarily commute with taking generic fibre. But it follows directly from the same explicit constructions: Let $S = \mathrm{Spa}(\mathcal{O}_K[[q^{1/p^\infty}]], \mathcal{O}_K[[q^{1/p^\infty}]])$ and consider the subspaces $S(|q|^{p^n} \leq |\varpi| \neq 0)$ which are rational because (q^n, ϖ) is open. As usual, one shows that since $\mathcal{O}_K[[q^{1/p^\infty}]]$ has ideal of definition (q, ϖ), the element $|q(x)|$ must be cofinal in the value group for any $x \in S$. This shows that

$$S^{ad}_\eta = \bigcup S(|q|^{p^n} \leq |\varpi| \neq 0).$$

Let (B_n, B_n^+) be the affinoid algebra corresponding to the rational subspace $S(|q|^{p^n} \leq |\varpi| \neq 0)$, then since $q^{p^n}/\varpi \in B_n^+$, the ideal (q, ϖ) equals (ϖ) and the ring B_n^+ thus has the ϖ-adic topology. From this one deduces that $B_n^+ = \mathcal{O}_K\langle q^{1/p^\infty}/\varpi^{1/np^\infty}\rangle$ and thus the spaces $S(|q|^{p^n} \leq |\varpi| \neq 0)$ and $D(|q|^{p^n} \leq |\varpi| \neq 0)$ coincide. □

Remark 3.6. Note that D_∞ is not affinoid, even though it is the generic fibre of an affine formal scheme. This is not a contradiction to the equivalence $K-\mathrm{Perf} = \mathcal{O}_K^{a\circ}-\mathrm{Perf}$, as one may think since the algebra $\mathcal{O}_K[[q^{1/p^\infty}]]$ looks very "perfectoid". But we have endowed it with the (p, q)-adic topology, and thus it isn't $\mathcal{O}_K^{a\circ}$-perfectoid. (Of course, suppressing the q-adic topology would make it perfectoid, but would also change the adic generic fibre).

Definition 3.7. The origin in D_∞ is a closed point $x : \mathrm{Spa}(K, \mathcal{O}_K) \to D_\infty$. Removing this point, we obtain a space $\mathring{D}_\infty := D_\infty \setminus \{x\}$ for which $\mathring{D}_\infty \sim \varprojlim_{q \mapsto q^p} \mathring{D}$.

We are now ready to discuss cusps at infinite level and the corresponding Tate curves.

Theorem 3.8. *Fix any cusp c of $\mathcal{X}^*$, we denote by the same later the cusps above it in the anticanonical tower.*

1. *The cusp morphisms $f(c)_n : D \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ in the limit give rise to an open immersion of perfectoid spaces $f(c)_\infty : D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$.*
2. *The following induced diagram is Cartesian:*

$$\begin{array}{ccc} D_\infty & \xhookrightarrow{f(c)_n} & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \\ \downarrow & & \downarrow \\ D & \xhookrightarrow{f(c)_\infty} & \mathcal{X}^*(\epsilon) \end{array}$$

3. *The morphism $f(c)_\infty$ restricts to an open immersion of perfectoid spaces*

$$f(c) : \mathring{D}_\infty \hookrightarrow \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a.$$

4. *Let (R, R°) be a perfectoid $(K, \mathcal{O}_K)$-algebra. Then the set*

$$\mathring{D}_\infty(R, R^\circ) \subseteq \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a(R, R^\circ)$$

is in functorial bijection with the set of pairs $(E_q, (q^{1/p^n})_{n\in\mathbb{N}})$ where E_q is a Tate curve over R for $q \in R$ a topologically nilpotent unit, and where $(q^{1/p^n})_{n\in\mathbb{N}}$ is a compatible system of p^n-th roots of q, determining an anticanonical $\Gamma_0(p^\infty)$-structure on E_q.

Proof. The existence of the morphism $D_\infty \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ follows from Proposition 2.10, Proposition 3.5 and the universal property of the perfectoid tilde-limit. Parts 1 to 3 of the Theorem now follow using [12], Propositions 2.4.3 from the fact that the squares in diagram (5) are all Cartesian.

The moduli interpretation of $\mathring{D}_\infty$, also follows from diagram (5). By Corollary 2.10 the (R, R°)-points of $\mathring{D} \to X^*_{\Gamma_0(p^n)}(\epsilon)_a$ correspond to Tate curves $Tate(q)$ over R with topologically nilpotent parameter q and a choice of p^n-th root q^{1/p^n} of q, in a way compatible with the forgetful morphisms. In particular the choices of q^{1/p^n} are compatible via $q \mapsto q^p$ in the tower. Since (R, R°) is perfectoid, we see from Proposition 3.5 that $\mathring{D}_\infty(R, R^\circ) = \varprojlim_{q\mapsto q^p} \mathring{D}(R, R^\circ)$. This shows that $\mathring{D}_\infty(R, R^\circ)$ corresponds to Tate curves $Tate(q)$ with $q \in R$ topologically nilpotent together with a compatible system of p^n-th roots of q. □

We finish this section by two Lemmas on formal models, which will later be useful when we compare the modular curve to its tilt. Instead of constructing one formal model for D_∞, we work with a family of formal models of the affinoid perfectoid subspaces $D_\infty(|q|^{p^n} \leq |\varpi|)$:

Lemma 3.9. *The flat formal scheme $\mathfrak{D}_\infty(q/\varpi^{1/p^n}) := \mathrm{Spf}(\mathcal{O}_K\langle (q/\varpi^{1/p^n})^{1/p^\infty}\rangle, (\varpi))$ is a formal model of $D_\infty(|q|^{p^n} \leq |\varpi|)$. The natural inclusion $\mathcal{O}_K[[q^{1/p^\infty}]] \hookrightarrow \mathcal{O}_K\langle (q/\varpi^{1/p^n})^{1/p^\infty}\rangle$ induces a morphism of formal schemes*

$$\psi : \mathfrak{D}_\infty(q/\varpi^{1/p^n}) \to \mathrm{Spf}(\mathcal{O}_K[[q^{1/p^\infty}]], (\varpi, q))$$

whose adic generic fibre ψ^{ad}_η is the inclusion $D_\infty(|q|^{p^n} \leq |\varpi|) \subseteq D_\infty$ from Proposition 3.5.(2)

Proof. Its clear that $\mathfrak{D}_\infty(q/\varpi^{1/p^n})$ is a flat formal model for $D_\infty(q/\varpi^{1/p^n})$. To construct the map ψ we just need to observe that the natural inclusion of adic rings $\mathcal{O}_K[[q^{1/p^\infty}]] \hookrightarrow \mathcal{O}_K\langle(q/\varpi^{1/p^n})^{1/p^\infty}\rangle$ is continuous. ☐

We remark that on the level of adic spaces, $\mathfrak{D}_\infty(q/\varpi^{1/p^n})$ is just the open subspace of $\mathrm{Spf}(\mathcal{O}_K[[q^{1/p^\infty}]], (\varpi, q))$ defined by $|q| \leq |\varpi^{1/p^n}|$, but we prefer to work in formal schemes since we don't know whether the right hand side is sheafy.

Lemma 3.10. *The restriction of the morphisms of perfectoid spaces*

$$D_\infty(|q| \leq |\varpi|^{1/p^n}) \subseteq D_\infty \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$$

has a canonical formal model $\mathfrak{D}_\infty(q/\varpi^{1/p^n}) \to \mathfrak{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$.

Proof. This follows by taking the inverse limit of the diagram of formal schemes

$$\begin{array}{ccccc}
\mathfrak{D}(q/\varpi^{1/p^{n+1}}) & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K[[q]]) & \longrightarrow & \mathfrak{X}^*(p^{-(n+1)}\epsilon) \\
{\scriptstyle q\mapsto q^p}\big\downarrow & & {\scriptstyle q\mapsto q^p}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{F}} \\
\mathfrak{D}(q/\varpi^{1/p^n}) & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K[[q]]) & \longrightarrow & \mathfrak{X}^*(p^{-n}\epsilon)
\end{array}$$

over the anticanonical tower. In the limit this gives the desired morphism of formal schemes

$$\mathfrak{D}(q/\varpi^{1/p^{n+1}}) \to \mathrm{Spf}(\mathcal{O}_K[[q^{1/p^\infty}]]) \to \mathfrak{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$$

which on the adic generic fibre is $D_\infty \to \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ because it is determined by the morphisms at finite level by the universal property of the perfectoid tilde limit. ☐

3.2 Tate parameter spaces of $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$

Next, we discuss the Tate parameter spaces in the pro-étale forgetful map $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$. This is essentially a matter of pulling back results from finite level: Let

$$\mathcal{X}^*_{\Gamma_1(p^n)\cup\Gamma_0(p^\infty)}(\epsilon)_a := \mathcal{X}^*_{\Gamma_1(p^n)}(\epsilon)_a \times_{\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a} \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a.$$

As usual, we define the cusps of this space to be the preimage of the cusps at finite level.

Proposition 3.11. *Assume that $\Gamma^p = \Gamma(N)$. Let c_0 be any cusp of $\mathcal{X}^*(\epsilon)$.*

1. *The forgetful map $\mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a$ gives a one-to-one correspondence of the cusps of both spaces. In particular, the cusps of $\mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a$ over c correspond to the choice of a generator of $\langle q^N\rangle \subseteq Tate(q^{p^nN})[p^n]$.*

2. *For the associated Tate parameter space $D_\infty \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$, there is a canonical Cartesian diagram*

$$\begin{array}{ccc}
\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D_\infty & \hookrightarrow & \mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a \\
\big\| & & \big\downarrow \\
D_\infty & \hookrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a
\end{array}$$

3. *The following diagram is Cartesian*

$$\begin{array}{ccc}
\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D_\infty & \hookrightarrow & \mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a \\
\downarrow & & \downarrow \\
\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D & \hookrightarrow & \mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a
\end{array}$$

where the morphism on the bottom is the one from Proposition 2.11.3.

4. *For varying n, the following diagram is Cartesian*

$$\begin{array}{ccc}
\underline{(\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \times D_\infty & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^{n+1})\cap\Gamma_0(p^\infty)}(\epsilon)_a \\
\downarrow & & \downarrow \\
\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D_\infty & \longrightarrow & \mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a
\end{array}$$

where the morphism on the left is the projection.

Proof. Part 1 follows by base-change from the description of the cusps of $\mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a$ together with Lemma 3.1. Part 2 and 3 follow from Proposition 2.11.(3) and Theorem 3.8 via the commutative cube

$$\begin{array}{ccccccc}
 & & \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D & & \longrightarrow & & D \\
 & \nearrow & & & & \nearrow & \\
\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D_\infty & & \longrightarrow & & D_\infty, & & \\
 & & \downarrow & & & & \downarrow \\
 & & \mathcal{X}^*_{\Gamma_1(p^m)}(\epsilon)_a & & \longrightarrow & & \mathcal{X}^*_{\Gamma_0(p^m)}(\epsilon)_a \\
\downarrow & \nearrow & & & \downarrow & \nearrow & \\
\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a & & \longrightarrow & & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a & &
\end{array}$$

in which the bottom, top, back and right square are Cartesian.

Part 4 follows from a similar commutative cube using as faces the Cartesian cubes from Corollary 2.11.(2) and the diagram from 3. $\square$

We now take the limit $m \to \infty$ to get to the space $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$: The situation in the case of $\Gamma_1(p^\infty)$ turns out to be slightly different to the situation for all the other modular curves we had so far: While we still obtain Tate-parameter spaces $D_\infty \to \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ around each cusp, these morphisms are not open immersions anymore. Vaguely speaking, this is due to the topological phenomenon that there are "profinitely many cusps".

Definition 3.12. For any profinite group G, choose a system of finite groups G_i with $G = \varprojlim G_i$, then we define $\underline{G}$ to be the unique perfectoid space which is the perfectoid tilde limit $\underline{G} \sim \varprojlim \underline{G_i}$. This is independent of the choice of G_i up to unique isomorphism.

Explicitly, $\underline{G}$ is the affinoid perfectoid space

$$\underline{G} = \mathrm{Spa}(\mathrm{Map}_{\mathrm{cts}}(G, K), \mathrm{Map}_{\mathrm{cts}}(G_i, \mathcal{O}_K)).$$

Theorem 3.1. *Let c be a cusp of $\mathcal{X}^*(\epsilon)$ and let $D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ be the corresponding Tate parameter space.*

1. *In the limit, the open immersions* $\underline{(\mathbb{Z}/p^m\mathbb{Z})^\times} \times D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a$ *give rise to a* $\mathbb{Z}_p^\times$*-equivariant open immersion* $f_c : \underline{\mathbb{Z}_p^\times} \times D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ *that fits into a Cartesian diagram*

$$\begin{array}{ccc} \underline{\mathbb{Z}_p^\times} \times D_\infty & \longrightarrow & D_\infty \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a. \end{array}$$

2. *The closed immersion* $\underline{\mathbb{Z}_p^\times} \hookrightarrow \underline{\mathbb{Z}_p^\times} \times D_\infty$ *induced by the origin* $\mathrm{Spa}(K, \mathcal{O}_K) \to D_\infty$ *composes with* f_c *to a locally closed morphism* $\underline{\mathbb{Z}_p^\times} \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ *whose image can be identified with the cusps of* $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ *over* c.

3. *For any* $a \in \mathbb{Z}_p^\times$*, the compatible choice of* $q^{aN} \in \langle q^{p^mN}\rangle$ *as a basis of* $Tate(q^{p^mN})[p^m]$ *induces a map* $\mathring{D}_\infty \to \mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$ *which extends uniquely to a locally closed immersion*

$$D_\infty \to \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a.$$

This morphism coincides with the immersion $D_\infty \hookrightarrow \underline{\mathbb{Z}_p^\times} \times D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ *where the first morphism is the closed immersion induced by the point* $a \in \mathbb{Z}_p^\times$.

Proof. 1. By Proposition 3.11.(4) there is a tower of Cartesian diagrams

$$\begin{array}{ccccccc} \dots \longrightarrow \underline{(\mathbb{Z}/p^{n+1}\mathbb{Z})^\times} \times D_\infty & \longrightarrow & \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D_\infty & \longrightarrow & \dots & \longrightarrow & D_\infty \\ \big\downarrow & & \big\downarrow & & & & \big\downarrow \\ \dots \to \mathcal{X}^*_{\Gamma_1(p^{m+1})\cap\Gamma_0(p^\infty)}(\epsilon)_a & \to & \mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a & \to & \dots & \to & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a. \end{array}$$

In the limit, this has the perfectoid tilde-limit $\underline{\mathbb{Z}_p^\times} \times D_\infty \sim \varprojlim \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D_\infty$ by [**?**], Lemma 2.12. Since the vertical arrows in the diagram are all open immersions, we conclude that in the limit we obtain the desired Cartesian diagram.

2. This follows from the fact that tilde-limits are limits on the level of topological spaces.

3. We first observe that the given data by Corollary 3.13 induces a map $\mathring{D} \to \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ which by the universal property of the tilde limit is uniquely determined by the compositions with the projections to finite level $\mathring{D} \to \mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a$. We know from Proposition **??**.2 that these morphisms are open immersions and extend over the cusps. More precisely, using Proposition **??**.1 and 2. we see that at level $\Gamma_1(p^m) \cap \Gamma_0(p^\infty)$, this extension is given by the composition

$$D_\infty \hookrightarrow \underline{(\mathbb{Z}/p^m\mathbb{Z})^\times} \times D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a$$

where the first map is the isomorphism onto the component corresponding to $a \bmod p^n$. In the limit we thus see that the induced morphism $D_\infty \to X^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ is given by

$$D_\infty \to \underline{\mathbb{Z}_p^\times} \times D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a$$

as described in 3. Since the point $\mathrm{Spa}(K, \mathcal{O}_K) \to \underline{\mathbb{Z}_p^\times}$ corresponding to a is a closed immersion, the morphism $D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ is the composition of a closed immersion with an open immersion, and thus is locally closed. ☐

3.3 Tate parameter spaces of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$

Finally, we look at the infinite level modular curve $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$. We first note that like before, we have the following moduli interpretation:

Corollary 3.13. *Let (R, R°) be a perfectoid $(K, \mathcal{O}_K)$-algebra. The set $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a(R, R^\circ)$ is in functorial bijection with isomorphism classes of triples (E, α_N, D, β) of an ϵ-nearly ordinary elliptic curve E over R, together with a Γ^p-structure α_N, and an isomorphism of p-divisible groups $\beta : (\mathbb{Q}_p/\mathbb{Z}_p)^2 \to E[p^\infty]$ over R (or equivalently an isomorphism $\mathbb{Z}_p^2 \to T_pE$) such that the restriction of β to the first factor is an anti-canonical $\Gamma_1(p^\infty)$-structure.*

Proof. This is an immediate consequence of Corollary 3.14 and [12], Proposition 2.4.5. □

Next, we look at the Tate parameter spaces in the pro-étale map $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$. As before, we do so by looking at the limit of the finite level morphisms

$$\mathcal{X}^*_{\Gamma(p^n)\cap\Gamma_0(p^\infty)} := \mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a \times_{\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a} \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a.$$

Corollary 3.14. *Let (R, R^+) be a perfectoid $(K, \mathcal{O}_K)$-algebra. Then the set of (R, R^+)-points $\mathcal{X}_{\Gamma(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a(R, R^+)$ is in functorial bijection with the set of isomorphism classes of ϵ-nearly ordinary elliptic curves E over R together with a choice of Γ^p-structure, a $\Gamma_0(p^\infty)_a$-structure $D \subseteq E[p^\infty]$ as well as an isomorphism $\alpha : (\mathbb{Z}/p^n\mathbb{N})^2 \to E[p^n]$ such that $\alpha(1,0)$ generates D_n.*

Proof. This is immediate from the moduli interpretation of $\mathcal{X}_{\Gamma(p^n)}$ and Corollary 3.2. □

We have the following description of the cusps of $\mathcal{X}^*_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a$, which we may define to be the complement of the open subspace $\mathcal{X}_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a$.

Proposition 3.15. *Assume that $\Gamma^p = \Gamma(N)$. Let c be any cusp of $\mathcal{X}^*(\epsilon)$.*

1. *The morphism $\mathcal{X}^*_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma(p^m)}(\epsilon)_a$ gives a bijection of cusps. In particular, the cusps of $\mathcal{X}^*_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a$ over c are parametrised by $\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$ where $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ corresponds to the ordered basis $(q^{dN/p^n}, \zeta_{p^n}^a q^{-bN/p^n})$ of $Tate(q^N)[p^n]$.*

2. *For the associated Tate parameter space $D_\infty \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$, there is a canonical Cartesian diagram*

$$\begin{array}{ccc} \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D_\infty & \xrightarrow{\varphi_\infty} & \mathcal{X}^*_{\Gamma(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a \\ \downarrow & & \downarrow \\ D_\infty & \hookrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \end{array}$$

3. *The following diagram is Cartesian:*

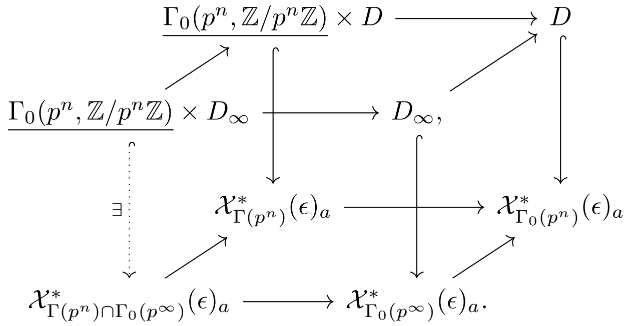
$$\begin{array}{ccc} \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D_\infty & \xrightarrow{\varphi_\infty} & \mathcal{X}^*_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a \\ \downarrow & & \downarrow \\ \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D & \xrightarrow{\varphi} & \mathcal{X}^*_{\Gamma(p^m)}(\epsilon)_a \end{array}$$

   where the morphism on the bottom is the morphism φ over c from Proposition 2.14.

4. *The following diagram is Cartesian, where the map on the left is given by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d$*

$$\begin{array}{ccc} \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D_\infty & \hookrightarrow & \mathcal{X}^*_{\Gamma(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a \\ \downarrow & & \downarrow \\ \underline{(\mathbb{Z}/p^n\mathbb{Z})^\times} \times D_\infty & \hookrightarrow & \mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a. \end{array}$$

Proof. Parts 1 to 3 follow from Proposition 2.14, Theorem 3.8 and the commutative cube



Part 5. follows from a similar commutative cube using the left square in Proposition 2.15 and Proposition 3.11. □

Lemma 3.16. *The morphisms from Proposition 3.15.2 for varying n give rise to the following tower of Cartesian diagrams*

$$\begin{array}{ccccccc} \ldots \to \underline{\Gamma_0(p^{n+1}, \mathbb{Z}/p^{n+1}\mathbb{Z})} \times D_\infty & \to & \underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D_\infty & \to & \ldots & \longrightarrow & D_\infty \\ \downarrow & & \downarrow & & & & \downarrow \\ \ldots \longrightarrow \mathcal{X}^*_{\Gamma(p^{m+1})\cap\Gamma_0(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a & \longrightarrow & \ldots & \to & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \end{array}$$

where the map on top is induced by the reduction $\Gamma_0(p^{n+1}, \mathbb{Z}/p^{n+1}\mathbb{Z}) \to \Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$.

Proof. This follows from Proposition 3.15.2 and Corollary **??** □

Definition 3.17. Let $\Gamma_0(p^\infty) = \Gamma_0(p^\infty, \mathbb{Z}_p)$ be the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ of matrices of the form $\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right)$. This is a profinite group because via $\mathrm{GL}_2(\mathbb{Z}_p) = \varprojlim \mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ we have

$$\Gamma_0(p^\infty) = \varprojlim_n \Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})$$

We are now ready to prove the main result of this section, namely a description of the cusps of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$. For the statement, let us briefly recall that the universal Tate curve over D_∞ is given by $Tate(q^N)$, in contrast to the situation at finite level $\Gamma(p^n)$ where the universal Tate curve is $Tate(q^{p^n N})$. For any n we have a canonical basis for $Tate(q^{p^n N})[p^n]$ given by $(q^{1/p^n}, \zeta_{p^n})$. In particular, we have a canonical basis of the Tate module $T_p Tate(q^N)$ given by the compatible system $(q^{N/p^n})_{n\in\mathbb{N}}$ that we denote by q^{N/p^∞} and the compatible system $(\zeta_{p^n})_{n\in\mathbb{N}}$ that we denote by ζ_{p^∞}.

Theorem 3.18. *Let c be a cusp of* $\mathcal{X}^*(\epsilon)$.

1. *In the limit, the open immersions* $\underline{\Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z})} \times D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a$ *give rise to an open immersion*

$$\underline{\Gamma_0(p^\infty)} \times D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a.$$

2. *The image of the closed immersion* $\underline{\Gamma_0(p^n,\mathbb{Z}/p^n\mathbb{Z})} \hookrightarrow \underline{\Gamma_0(p^n,\mathbb{Z}/p^n\mathbb{Z})} \times D_\infty$ *defined by* $q \mapsto 0$ *is precisely the subset of cusps of* $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ *lying over c via* $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \to \mathcal{X}^*$.

3. *For any* $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma_0(p^\infty)$*, the cusp of* $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ *obtained by specialising at* γ *is the one corresponding to the isomorphism* $\mathbb{Z}_p^2 \to T_pTate(q^N)$ *defined by the basis* $(q^{dN/p^\infty}, \zeta^a_{p^\infty} q^{-bN/p^\infty})$ *of* $T_pTate(q^N)$.

4. *The following commutative diagram is a tower of Cartesian squares*

$$\begin{array}{ccccccc}
\underline{\Gamma_0(p^\infty)} \times D_\infty & \longrightarrow & \underline{\mathbb{Z}_p^\times} \times D_\infty & \longrightarrow & D_\infty & \longrightarrow & D \\
\big\downarrow & & \big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a & \longrightarrow & X^*_{\Gamma_1(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*(\epsilon).
\end{array}$$

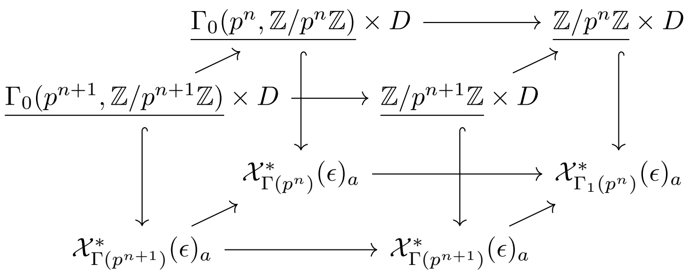
where the morphism on the top left is $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto d$.

Proof. 1. This is a consequence of Lemma 3.16, and Lemma 2.12 in [?].

2. This follows from Proposition 3.15.1 and Lemma 3.16, together with the fact that $|\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a| = \varprojlim |\mathcal{X}^*_{\Gamma(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)_a|$ by definition of the tilde-limit.

3. Follows from Proposition 3.15.3 and Lemma 3.16.

4. It suffices to show that the left square commutes, since we have already shown that the other squares are Cartesian in Theorem **??** and Theorem 3.8. But this is a consequence of 2.15 in the limit over n: By Lemmas **??** and **??** and Proposition 2.14, the following is a commutative cube with all vertical squares Cartesian:



where the diagonal morphisms on top are given by reduction on the first component, and $q \mapsto q^p$ on the second component. In the limit over n, this shows that the left square in 4. is commutative. That it is Cartesian follows from the fact that pullbacks commute with perfectoid tilde-limits. □

We note the following easy consequence (the analogue of this for Siegel moduli spaces for genus $g > 1$ is proved in the proof of [11], Lemma III.2.35).

Corollary 3.19. *For any* $n \in \mathbb{N} \cup \{\infty\}$*, our choice of* ζ_p^∞ *induces a canonical isomorphism*

$$\mathcal{X}^*_{\Gamma(p^n)}(0)_a = \bigsqcup_{\Gamma(p^n)/\Gamma_1(p^n)} \mathcal{X}^*_{\Gamma_1(p^n)}(0)_a$$

Proof. For $n = \infty$, there is away from the cusps a canonical splitting induced by $T_pE = T_pC \times T_pD$ and the canonical isomorphism $T_pC = T_pD^\vee$ induced from the Weil pairing. On Tate parameter spaces, one checks that this splitting is given by $\underline{\mathbb{Z}_p^\times} \times \mathring{D}_\infty \to \underline{\Gamma_0(p^\infty)} \times \mathring{D}_\infty$, $(a,q) \mapsto (\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right), q)$, which clearly extends over the boundary. Similarly for $n < \infty$. □

3.4 The action of $\Gamma_0(p)$ on the cusps of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$

Finally in this section, we discuss the action of the full action on the Tate parameter spaces at infinite level. We first recall which group acts at infinite level:

Definition 3.20. Let $\Gamma_0(p) = \Gamma_0(p, \mathbb{Z}_p)$ be the subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ of matrices of the form $\left(\begin{smallmatrix} * & * \\ c & * \end{smallmatrix}\right)$ with $c \equiv 0 \mod p$. This is a profinite group with $\Gamma_0(p) = \varprojlim_n \Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$.

Since the $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$-action on each $\mathcal{X}^*_{\Gamma(p^n)}$ restricts to a $\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$-action on the subspace $\mathcal{X}^*_{\Gamma(p^n)}(\epsilon)_a$ as is evident from the moduli description, we see that the $\mathrm{GL}_2(\mathbb{Z}_p)$-action on $\mathcal{X}^*_{\Gamma(p^\infty)}$ restricts to a $\Gamma_0(p)$-action on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$.

Theorem 3.21. *Over any cusp c of $\mathcal{X}^*$, the $\Gamma_0(p)$-action on $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ restricts to an action on $\varphi_\infty : \underline{\Gamma_0(p^\infty)} \times D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ where it can be described as follows: Equip $\underline{\Gamma_0(p)} \times D$ with a right action by $p\mathbb{Z}_p$ via $(\gamma, q^{1/p^m}) \mapsto (\gamma \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right), \zeta_{p^m}^{h/N} q^{1/p^m})$ for $h \in p\mathbb{Z}_p$, then*

$$(\underline{\Gamma_0(p)} \times D_\infty)/p\mathbb{Z}_p = \underline{\Gamma_0(p^\infty)} \times D_\infty$$

as sheaves and the left action of $\Gamma_0(p)$ is the one induced by letting $\Gamma_0(p)$ act on the first factor of $\underline{\Gamma_0(p)} \times D_\infty$. Explicitly, in terms of any $\gamma_1 \in \Gamma_0(p)$, the action is given by

$$\begin{aligned} \gamma_1 : \underline{\Gamma_0(p^\infty)} \times D_\infty &\xrightarrow{\sim} \underline{\Gamma_0(p^\infty)} \times D_\infty \\ \gamma_2, q^{1/p^m} &\mapsto \begin{pmatrix} \det(\gamma_3)/d_3 & b_3 \\ 0 & d_3 \end{pmatrix}, \zeta_{p^m}^{-\frac{c_3}{d_3 N}} q^{1/p^m}. \end{aligned}$$

where $\gamma_3 = \left(\begin{smallmatrix} a_3 & b_3 \\ c_3 & d_3 \end{smallmatrix}\right) := \gamma_1 \cdot \gamma_2$.

Proof. That the action restricts to an action on $\underline{\Gamma_0(p^\infty)} \times D_\infty$ is a consequence Proposition 2.17 in the limit over n. The same argument gives the explicit formula.

Let us explain what we mean by the equality $(\underline{\Gamma_0(p)} \times D_\infty)/p\mathbb{Z}_p = \underline{\Gamma_0(p^\infty)} \times D_\infty$. One way to see this is as an isomorphism of diamonds, but for simplicity we may just work with the category of sheaves on the category $\mathbf{Perf}_K$. In particular, the quotient $(\underline{\Gamma_0(p)} \times D_\infty)/p\mathbb{Z}_p$ is to be taken in the category of sheaves on $\mathbf{Perf}_K$.

The isomorphism can then be constructed as a limit of the isomorphisms at finite level: One checks that the following diagram commutes:

$$\begin{array}{ccccccc} \underline{\Gamma_0(p, \mathbb{Z}/p^{n+1}\mathbb{Z})} \times D & \to & \underline{\Gamma_0(p^{n+1}, \mathbb{Z}/p^{n+1}\mathbb{Z})} \times D & \quad & \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), q & \mapsto & \begin{pmatrix} \det(\gamma_3)/d_3 & b_3 \\ 0 & d_3 \end{pmatrix}, \zeta_{p^{n+1}}^{-c_3/d_3 N} q \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \underline{\Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})} \times D & \longrightarrow & \Gamma_0(p^n, \mathbb{Z}/p^n\mathbb{Z}) \times D & & \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), q^p & \mapsto & \begin{pmatrix} \det(\gamma_3)/d_3 & b_3 \\ 0 & d_3 \end{pmatrix}, \zeta_{p^n}^{-c_3/d_3 N} q^p \end{array}$$

(to avoid confusion we emphasize that on the right we are describing the maps in terms of *points*, and therefore the lower horizontal is indeed given by multiplication by $\zeta_{p^n}^{-c_3/d_3 N}$, whereas on the level of *functions* it sends $q^p \mapsto \zeta_{p^{n-1}}^{-c_3/d_3 N} q^p$). When we endow the spaces on the left with the $p\mathbb{Z}_p$ actions via the reductions $p\mathbb{Z}_p \to p\mathbb{Z}/p^{n+1}\mathbb{Z}$ and $p\mathbb{Z}_p \to p\mathbb{Z}/p^n\mathbb{Z}$ respectively, we moreover see that the vertical morphism on the left is equivariant under the action of $p\mathbb{Z}_p$. The diagram is moreover equivariant for the $\Gamma_0(p)$-action on the left via the reductions $\Gamma_0(p) \to \Gamma_0(p, \mathbb{Z}/p^{n+1}\mathbb{Z})$ and $\Gamma_0(p) \to \Gamma_0(p, \mathbb{Z}/p^n\mathbb{Z})$ respectively. In the limit we therefore obtain a $p\mathbb{Z}_p$-invariant morphism

$$\underline{\Gamma_0(p)} \times D_\infty \to \underline{\Gamma_0(p^\infty)} \times D_\infty$$

which is equivariant for the $\Gamma_0(p)$-action on the left of both sides.

We are left to see that this is a quotient map of sheaves for the action of $p\mathbb{Z}_p$, which is easy to check on points: On the level of sets, we have $\Gamma_0(p)/p\mathbb{Z}_p = \Gamma_0(p^\infty)$ with a natural set-theoretic section given by the inclusion map, giving a bijection $\Gamma_0(p) = \Gamma_0(p^\infty) \times p\mathbb{Z}_p$. For any perfectoid K-algebra (R, R^+) for which $|\operatorname{Spa}(R, R^+)|$ is connected we then have

$$\begin{aligned}(\underline{\Gamma_0(p^\infty)} \times D_\infty)(R, R^+) =& \Gamma_0(p^\infty) \times D_\infty(R, R^+) = (\Gamma_0(p) \times D_\infty(R, R^+))/p\mathbb{Z}_p \\ =& (\underline{\Gamma_0(p)} \times D_\infty)(R, R^+)/p\mathbb{Z}_p.\end{aligned}$$

where the second equality is via $\Gamma_0(p) = \Gamma_0(p^\infty) \times p\mathbb{Z}_p$. This shows that $\underline{\Gamma_0(p^\infty)} \times D_\infty$ indeed has the universal property of the quotient in the category of sheaves on $\mathbf{Perf}_K$. $\square$

Perhaps should also say something about classification of points at infinite level.

3.5 The Hodge-Tate period map on Tate parameter spaces

In this section we want to see what the Hodge-Tate map looks like on Tate parameter spaces.

Recall that over the ordinary locus, the Hodge-Tate map $T_pE \to \omega_E$ has kernel T_pC the Tate module of the canonical p-divisible subgroup, and thus the Hodge filtration is given by $T_pC \to T_pE$. In particular, this means that

$$\pi_{HT}(\mathcal{X}^*_{\Gamma(p^\infty)}(0)) \subseteq \mathbb{P}^1(\mathbb{Z}_p).$$

When we further restrict to the anticanonical locus, the image lies in the points of the form $(a:1) \in \mathbb{P}^1(\mathbb{Z}_p)$ with $a \in \mathbb{Z}_p$. In particular, when we denote by $B_1(0) \subseteq \mathbb{P}^1(\mathbb{Z}_p)$ the ball of radius 1 inside the canonical chart $\mathbb{A}^1 \subseteq \mathbb{P}^1$ around $(0:1)$, the Hodge-Tate period map restricts to

$$\pi_{HT}(\mathcal{X}^*_{\Gamma(p^\infty)}(0)_a) \subseteq B_1(0) \subseteq \mathbb{P}^1(\mathbb{Z}_p).$$

On $B_1(0)$ there is a canonical parameter z given by the coordinate $(z:1)$. We denote its pullback to $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ by $\mathfrak{z}$.

Proposition 3.22.

1. *Let $\underline{\mathbb{Z}_p^\times} \to B_1(0)$ be the natural morphism given by $a \mapsto a$. Consider the morphism*

$$\varphi : \underline{\Gamma_0(p^\infty)} \times D_\infty \to \underline{\mathbb{Z}_p}, \quad \left(\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right), x\right) \mapsto b/d.$$

   Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \in \Gamma_0(p^\infty)$ and any cusp, the following diagram is commutative:

$$\begin{array}{ccccccc} D_\infty & \xrightarrow{q\mapsto(\gamma,q)} & \underline{\Gamma_0(p^\infty)} \times D_\infty & \hookrightarrow & \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a & \longrightarrow & \mathcal{X}^*_{\Gamma(p^\infty)} \\ \downarrow & & \downarrow \varphi & & \downarrow & & \downarrow \pi_{HT} \\ \operatorname{Spa}(K, \mathcal{O}_K) & \xrightarrow{b/d} & \underline{\mathbb{Z}_p} & \hookrightarrow & B_1(0) & \xhookrightarrow{a\mapsto(a:1)} & \mathbb{P}^1 \end{array}$$

2. *The natural parameter $\mathfrak{z}$ restricts on $\underline{K_0(p^\infty)} \times D_\infty$ to the function*

$$\mathfrak{z}|_{\underline{K_0(p^\infty)} \times D_\infty} = \Big(\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) \mapsto b/d \Big) \in \operatorname{Map}_{\mathrm{cts}}(K_0(p^\infty), \mathcal{O}(D_\infty)).$$

We deduce this from the following Lemma:

Lemma 3.23. *Let $F : D_\infty \to \mathbb{A}^1_K$ be function such that $F = a$ is constant on $(C, \mathcal{O}_C)$-points. Then the corresponding function in $\mathcal{O}(D_\infty)$ is given by the constant $a \in K \subseteq \mathcal{O}(D_\infty)$.*

Proof. It suffices to prove this for any of the spaces $D_\infty(|q| \le \varpi^n)$. After rescaling, we are reduced to showing the Lemma for D_∞ replaced by $\mathrm{Spa}(K\langle q^{1/p^\infty}\rangle, \mathcal{O}_K\langle q^{1/p^\infty}\rangle)$. One can now argue like in the classical proof of the maximum principle: Let $f \in K\langle q^{1/p^\infty}\rangle$, $f = \sum_{m\in\mathbb{Z}[1/p]\ge 0} a_m q^m$ be the function corresponding to F. We need to prove that if $f((x^{1/p^i})_{i\in\mathbb{N}})$ is constant for all $(x^{1/p^i})_{i\in\mathbb{N}} \in \varprojlim_{x\mapsto x^p} \mathcal{O}_C$ then f is constant. After subtracting by a_0, we may assume that $f(x) = 0$ for all $x \in \mathcal{O}_C$.

Suppose that $f \neq 0$. The convergence condition on coefficients assures that the supremum $\sup_{m\in\mathbb{Z}[1/p]} |a_m| > 0$ is attained and after dividing through by a_m for which $|a_m|$ is maximal, we may assume that $|f| = \max_{m\in\mathbb{Z}[1/p]} |a_m| = 1$. In this case, consider the map

$$r : \mathcal{O}_K\langle q^{1/p^\infty}\rangle \to k[q^{1/p^n} | n \in \mathbb{N}]$$

that we get from reducing by the maximal ideal $\mathfrak{m} \subseteq \mathcal{O}_K$. After replacing $q \mapsto q^{p^k}$ we may assume that $r(f) \in k[q]$. Since $\mathcal{O}_C$ is perfectoid, the projection map $\varprojlim \mathcal{O}_C \to \mathcal{O}_C \to k$ is surjective, and the assumption on f now implies that $r(f)$ is a non-zero polynomial in $k[q]$ which is $= 0$ for all $q \in \overline{k}$, a contradiction. ☐

proof of Proposition 3.22. By the Lemma it suffices to prove that for any $\gamma \in K_0(p^\infty)$ the morphism $D_\infty \xrightarrow{q\mapsto\gamma,q} \underline{K_0(p^\infty) \times D_\infty} \to \mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a \xrightarrow{\pi_{HT}} \mathbb{P}^1$ is constant on topological spaces with image b/d. To see this, we may use the moduli interpretation of π_{HT} on $(C, \mathcal{O}_C)$-points:

On the ordinary locus, it sends any $\mathbb{Z}_p^2 \to T_pE$ to point of $\mathbb{P}^1(\mathbb{Z}_p)$ defined by the line $T_pC \subseteq T_pE$ where C is the canonical p-divisible subgroup. By Theorem 3.18.3, any $(C, \mathcal{O}_C)$-point of $D_\infty \xrightarrow{q\mapsto\gamma,q} \underline{K_0(p^\infty) \times D_\infty}$ corresponds to a Tate curve E_q with basis of T_pE_q given by $(e_1, e_2) = (q^{dN/p^\infty}, \zeta^a_{p^\infty} q^{-bN/p^\infty})$. One checks that (using additive notation on T_pE)

$$be_1 + de_2 = q^{bdN/p^\infty} \zeta^{ad}_{p^\infty} q^{-dbN/p^\infty} = \zeta^{ad}_{p^\infty}$$

which spans the line $\langle \zeta_{p^\infty}\rangle = T_pC \subseteq T_pE$. Consequently, the image of (γ, q) under π_{HT} is

$$\pi_{HT}(\gamma, q) = (b : d) = (b/d : 1) \in \mathbb{Z}_p^\times \subseteq \mathbb{P}^1(\mathbb{Z}_p).$$

Using the Lemma, this shows that $\pi_{HT}(\gamma, -) : D_\infty \to \mathbb{P}^1$ is defined by the constant $b/d \in K \subseteq \mathcal{O}(D_\infty)$.

We conclude from this that the function $f \in \mathrm{Map}_{\mathrm{cts}}(K_0(p^\infty), \mathcal{O}(D_\infty))$ defined by $\pi_{HT} : \underline{K_0(p^\infty)} \times D_\infty \to B(0)$ evaluates at γ to $f(\gamma) = b/d$. Since this is true for all $\gamma \in K_0(p^\infty)$, we see that f is given by a function in $\mathrm{Map}_{\mathrm{cts}}(K_0(p^\infty), \mathbb{Z}_p^\times) \subseteq \mathrm{Map}_{\mathrm{cts}}(K_0(p^\infty), \mathcal{O}(D_\infty))$. Consequently, π_{HT} factors through

$$\underline{K_0(p^\infty)} \times D_\infty \to \underline{\mathbb{Z}_p^\times}, \quad (\gamma, q) \mapsto b/d$$

as desired. The second part is then an immediate consequence by taking global sections. ☐

3.6 Tate parameter spaces of the modular curve at infinite level

As an immediate Corollary of the above, we can now consider the case of $\mathcal{X}^*_{\Gamma(p^\infty)}$. Recall that by the very construction in [11], this is the space $\mathrm{GL}_2(\mathbb{Q}_p)\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$ defined by glueing translates of $\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a$. We can thus deduce from our results so far:

Theorem 3.2. *1. Consider the right action of $\mathbb{Z}_p$ on the perfectoid space $\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty$ defined by $(\gamma, q) \cdot h \mapsto (\gamma \left(\begin{smallmatrix} 1 & 0 \\ h & 1 \end{smallmatrix}\right), \zeta_{p^\infty}^h q^{1/p^\infty})$. Then the quotient $(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p$ exists as an adic space. Let c be any cusp of $\mathcal{X}^*$. Then the pullback of the corresponding Tate parameter space $D \hookrightarrow \mathcal{X}^*$ along the projection $\mathcal{X}^*_{\Gamma(p^\infty)} \to \mathcal{X}^*$ is of the form*

$$\begin{array}{ccc} (\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^*_{\Gamma(p^\infty)} & \longrightarrow & \mathcal{X}^* \end{array}$$

where the morphism on top is projection to the second factor. The morphism on the left is canonical after a choice of ζ_{p^∞} and is then $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariant for the natural left action on $(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p$ induced by letting $\mathrm{GL}_2(\mathbb{Z}_p)$ act on the first factor.

2. The map $\pi_{HT} : \mathcal{X}^*_{\Gamma(p^\infty)} \to \mathbb{P}^1$ restricts to $(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p \to \underline{\mathbb{P}^1(\mathbb{Z}_p)}$ given by

$$(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), q) \mapsto (b : d).$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} (\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p & \longrightarrow & \underline{\mathbb{P}^1(\mathbb{Z}_p)} \\ \big\downarrow & & \big\downarrow \\ \mathcal{X}^*_{\Gamma(p^\infty)} & \xrightarrow{\pi_{HT}} & \mathbb{P}^1. \end{array}$$

Proof. To see that $(\underline{\mathrm{GL}_2(\mathbb{Z}_p)} \times D_\infty)/\mathbb{Z}_p$ exists as an adic space, we simply note that we can construct it as $\mathrm{GL}_2(\mathbb{Z}_p) \cdot (\underline{K_0(p^\infty)} \times D_\infty)/\mathbb{Z}_p$, that is by glueing copies of $(\underline{K_0(p^\infty)} \times D_\infty)/\mathbb{Z}_p$ using the action of $\mathrm{GL}_2(\mathbb{Z}_p)$. Since we may without loss of generality replace $\mathcal{X}^*_{\Gamma(p^\infty)}$ by $\mathcal{X}^*_{\Gamma(p^\infty)}(0)$, and the latter is simply $\mathrm{GL}_2(\mathbb{Z}_p) \cdot \mathcal{X}^*_{\Gamma(p^\infty)}(0)_a$, the first part then follows from translating the Cartesian diagram from Theorem 3.2 by the $\mathrm{GL}_2(\mathbb{Z}_p)$-action. The second part follows from Proposition 3.22 by $\mathrm{GL}_2(\mathbb{Z}_p)$-equivariance of π_{HT}. ☐

4 Modular curves in characteristic p

We now switch to analytic moduli spaces in characteristic p. More precisely, we work over $(K^\flat, \mathcal{O}_{K^\flat})$. Let X' be the tame level Γ^p modular curve over $\mathcal{O}_{K^\flat}$, with generic fibre $X'_{K^\flat}$ over $K^\flat$. We denote by X'^* and $X'^*_{K^\flat}$ the minimal compactifications. We denote by $X'_{ord} \subseteq X'$ the affine open subscheme where the Hasse invariant Ha is invertible. Similarly, one defines $X'^*_{ord} \subseteq X'^*$ which is also affine open.

Let $\mathfrak{X}'$ be the $\varpi^\flat$-adic completion of X' and let $\mathcal{X}'$ be the analytification of $X'_{K^\flat}$, similarly for the compactifications. Like in the characteristic 0 case, for $0 \leq \epsilon < 1/2$ we denote by $\mathcal{X}'^*(\epsilon)$ the open subspace of $\mathcal{X}'^*$ where $|\,\mathrm{Ha}\,| \geq |\varpi|^\epsilon$. Like before, there is a canonical formal model $\mathfrak{X}'^*(\epsilon) \to \mathfrak{X}'^*$. For any adic space $\mathcal{Y} \to \mathcal{X}'^*$ we write $\mathcal{Y}(\epsilon) := \mathcal{Y} \times_{\mathcal{X}'^*} \mathcal{X}'^*(\epsilon)$. In particular, there is the open subspace $\mathcal{X}'(\epsilon) \subseteq \mathcal{X}'^*(\epsilon)$. We recall that while the elliptic curves parametrised by this space might have good supersingular *reduction*, the condition on the Hasse invariant ensures that *generically*, these elliptic curves are ordinary. In other words, $\mathcal{X}'(\epsilon) \subseteq \mathcal{X}'_{ord}$.

4.1 Igusa curves

Let us recall that in characteristic p one has the Igusa moduli problem:

Definition 4.1 ([9], Definition 12.3.1)**.** Let S be a scheme of characteristic p and let E be an elliptic curve over S. Consider the Verschiebung morphism $\ker V^n : E^{(p^n)} \to E$. An Igusa structure on E is a morphism $\phi : \mathbb{Z}/p^n\mathbb{Z} \to E^{(p^n)}(S)$ that is a Drinfeld generator of $\ker V^n$, that is such that the Cartier divisor $\sum_{a \bmod p^n}[\phi(a)] \subseteq E^{(p^n)}$ coincides with $\ker V^n$.

The Igusa problem $[Ig(p^n)]$ is the moduli problem defined by the functor sending $E|S$ to the set of Igusa structures on E. In case that $E|S$ is ordinary, the group scheme $\ker V^n$ is étale and an Igusa structure on S then always exists after an étale extension of S. In particular, in this situation a $\mathrm{Ig}(p^n)$-structure is the same as an isomorphism $\underline{\mathbb{Z}/p^n\mathbb{Z}} \to C_n^\vee$, which by Cartier-duality is the same as an isomorphism $C_n \to \mu_{p^n}$.

For any $n \geq 0$, the Igusa problem $[\mathrm{Ig}(p^n)]$ is relatively representable, finite and flat of degree $\varphi(p^n)$ on $Ell|\mathbb{F}_p$ by [9], Theorem 12.6.1. In particular, the simultaneous moduli problem $[\mathrm{Ig}(p^n), \Gamma^p]$ is representable by a moduli scheme $X_{\mathbb{F}_p,\mathrm{Ig}(p^n)}$ over $\mathbb{F}_p$. The forgetful map $f : X'_{\mathbb{F}_p,\mathrm{Ig}(p^n)} \to X'_{\mathbb{F}_p}$ is finite and flat, and is an étale $(\mathbb{Z}/p^n\mathbb{Z})^\times$-torsor over the ordinary locus $X'^{ord}_{\mathbb{F}_p} \subseteq X'_{\mathbb{F}_p}$ with group $(\mathbb{Z}/p^n\mathbb{Z})^\times$. There is moreover a finite flat forgetful map $X'_{\mathbb{F}_p,\mathrm{Ig}(p^{n+1})} \to X'_{\mathbb{F}_p,\mathrm{Ig}(p^n)}$. One defines by normalisation a compactification $X^*_{\mathbb{F}_p,\mathrm{Ig}(p^n)}$. The morphism $f : X'_{\mathbb{F}_p,\mathrm{Ig}(p^n)} \to X'_{\mathbb{F}_p}$ then extends to a map

$$f : X'^*_{\mathbb{F}_p,\mathrm{Ig}(p^n)} \to X'^*_{\mathbb{F}_p}$$

which is still finite Galois with group $(\mathbb{Z}/p^n\mathbb{Z})^\times$ over the ordinary locus.

For any morphism $\mathrm{Spec}(\mathbb{F}_q((q))) \to X'_{\mathbb{F}_q}$ corresponding to a choice of $\Gamma(N)$-structure on $Tate(q^N)$, any choice of isomorphism $\mu_{p^n} \xrightarrow{\sim} \mu_{p^n} \subseteq Tate(q^N)[p^n]$ induces a morphism $\mathrm{Spec}(\mathbb{F}_q((q))) \to X'_{\mathbb{F}_q,Ig(p^n)}$ making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Spec}(\mathbb{F}_q((q))) & \longrightarrow & X'_{\mathbb{F}_q,Ig(p^n)} \\ \| & & \downarrow \\ \mathrm{Spec}(\mathbb{F}_q((q))) & \longrightarrow & X'_{\mathbb{F}_q} \end{array}$$

In particular, over any cusp of $X'^*_{\mathbb{F}_q}$ there are precisely $\varphi(p^n)$ disjoint cusps of $X'^*_{\mathbb{F}_q,\mathrm{Ig}(p^n)}$.

We denote by $X'_{\mathrm{Ig}(p^n)}$ the base change of $X'_{\mathbb{F}_p,\mathrm{Ig}(p^n)}$ to $\mathcal{O}_K$. Like for X' one defines by completion formal schemes $\mathfrak{X}'_{\mathrm{Ig}(p^n)}$ and $\mathfrak{X}'^*_{\mathrm{Ig}(p^n)}$ as well as analytifications $\mathcal{X}'_{\mathrm{Ig}(p^n)}$ and $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}$, as well as open subspaces $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)$. Since $\mathcal{X}'^*(\epsilon) \subseteq \mathcal{X}'^*_{ord}$, the morphism $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon) \to \mathcal{X}'^*(\epsilon)$ is a finite étale $\mathbb{Z}/p^n\mathbb{Z}$-torsor. Like in the case of characteristic 0, these spaces represent the obvious adic moduli functors by Lemma **??**, using that X' is affine.

Definition 4.2. We call Igusa tower the inverse system of forgetful morphisms

$$\cdots \to \mathcal{X}'^*_{\mathrm{Ig}(p^{n+1})}(\epsilon) \to \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon) \to \cdots \to \mathcal{X}'^*(\epsilon).$$

Note that all the transition maps in this inverse system are finite étale.

4.2 Tate parameter spaces for Igusa curves

Next, we wish to analyse the analytic situation at the cusp. Let c be a cusp of $X'^*_{\mathrm{Ig}(p^n)}$ and let $\mathrm{Spf}(\mathcal{O}_{K^\flat}[\zeta_N][[q]]) \to X'^*_{\mathrm{Ig}(p^n)}$ be the completion along c. Upon ϖ-adic completion this gives a morphism

$$\mathrm{Spf}(\mathcal{O}_{K^\flat}[\zeta_N][[q]], (q, \varpi)) \to \mathfrak{X}'^*_{\mathrm{Ig}(p^n)}.$$

Denote by $f(c)_\eta : D' \to \mathcal{X}'^*_{\mathrm{Ig}(p^n)}$ the generic fibre. Here D' is the open subspace of the closed disc $\mathrm{Spa}(K^\flat\langle q\rangle, \mathcal{O}_{K^\flat}\langle q\rangle)$ over $K^\flat$ defined by $|q| < 1$. Then Conrad's Berthelot generic fibre construction gives the analogue of Theorem 2.3 for Igusa curves:

Proposition 4.3. *The morphism $f(c)_\eta : D' \to \mathcal{X}'^*_{\mathrm{Ig}(p^n)}$ of rigid spaces is an open immersion that identifies D' with an open neighbourhood of the cusp c.*

Lemma 4.4. *Denote by w the map of locally ringed spaces $w : D' \to \mathrm{Spec}(\mathbb{F}_q[[q]] \otimes \mathcal{O}_{K^\flat})$ induced by the natural inclusion $\mathbb{F}_q[[q]] \otimes \mathcal{O}_{K^\flat} \hookrightarrow \mathcal{O}_{D'}(D')$. Then the following diagram of locally ringed spaces commutes:*

$$\begin{array}{ccc}
\mathrm{Spec}(\mathbb{F}_q[[q]] \otimes \mathcal{O}_{K^\flat}) & \xrightarrow{c} & X'^*_{\mathrm{Ig}(p^n)} \\
{\scriptstyle w}\uparrow & & \uparrow \\
D & \xrightarrow{c_\eta} & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}.
\end{array}$$

Proof. Exactly like for Lemma 2.5. ☐

Proposition 4.5. *Let c_0 be a cusp of $\mathcal{X}'^*$.*

1. *There is a canonical Cartesian diagram*

$$\begin{array}{ccc}
\underline{\mathbb{Z}/p^n\mathbb{Z}} \times D' & \longrightarrow & D' \\
\downarrow & & \uparrow \\
\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon) & \longrightarrow & \mathcal{X}'^*(\epsilon)
\end{array}$$

   where the map on the right is the union of the maps $f(c)_\eta$ for all cusps c over c_0.

2. *There is a canonical Cartesian diagram, where the morphism on top is the projection*

$$\begin{array}{ccc}
\underline{\mathbb{Z}/p^{n+1}\mathbb{Z}} \times D' & \longrightarrow & \underline{\mathbb{Z}/p^n\mathbb{Z}} \times D' \\
\downarrow & & \downarrow \\
\mathcal{X}'^*_{\mathrm{Ig}(p^{n+1})}(\epsilon) & \longrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon).
\end{array}$$

Proof. This can be proved like in 2.11, using Lemma **??** and Lemma 4.4. ☐

Corollary 4.6. *Let c be the cusp over c_0 corresponding to $a \in \mathbb{Z}/p^n\mathbb{Z}$. For any honest adic space S over $K^\flat$, a morphism $S \to \mathcal{X}'_{\mathrm{Ig}(p^n)}$ factors through $f(c) : D' \hookrightarrow \mathcal{X}'^*_{\mathrm{Ig}(p^n)}$ if and only if it corresponds to a Tate curve over $\mathcal{O}(S)$ with topologically nilpotent parameter q and with $\mathrm{Ig}(p^n)$-structure given by $\mu_{p^n} \xrightarrow{a} \mu_{p^n} \subseteq Tate(q^N)$. Equivalently, by duality, the $\mathrm{Ig}(p^n)$-structure is given by the choice of basis q^a of $\langle q\rangle \subseteq Tate(q^{p^n N})$.*

Proof. This can be deduced from the Lemma and the Proposition like in Corollary 2.6. ☐

Lemma 4.7. *For any $n \in \mathbb{N}$, the following diagrams are Cartesian:*

$$(1)\quad \begin{array}{ccc}
D' & \hookrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(p^{-1}\epsilon) \\
\downarrow{\scriptstyle q\mapsto q^p} & & \downarrow{\scriptstyle F_{rel}} \\
D' & \hookrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon).
\end{array}
\qquad
(2)\quad \begin{array}{ccc}
\mathcal{X}'^*_{\mathrm{Ig}(p^{n+1})}(p^{-1}\epsilon) & \longrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(p^{-1}\epsilon) \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} \\
\mathcal{X}'^*_{\mathrm{Ig}(p^{n+1})}(\epsilon) & \longrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon).
\end{array}$$

Proof. The diagrams commute by functoriality of Frobenius. We are thus left to see they are Cartesian

1. It suffices to check this on $(C, \mathcal{O}_C)$-points because the horizontal morphisms are open immersions: It is clear that the cusps correspond, since these do not depend on ϵ and $q \mapsto q^p$ sends the origin to the origin. Away from the cusps, we can check that the diagram is Cartesian using the moduli description from Corollary 4.6.

2. We can see this by comparing $\mathcal{X}'^*_{\mathrm{Ig}(p^{n+1})}(p^{-1}\epsilon)$ to the fibre product in the category of finite étale adic spaces over $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(p^{-1}\epsilon)$, and using [7] Example 1.6.6.(ii). Alternatively, we can check this on moduli interpretations, using that the relative Frobenius maps $\ker V^n|E^{(p^n)}$ isomorphically onto $\ker V^n|E^{(p^{n+1})}$ in the case that E is ordinary. Then look at Tate parameter spaces to extend over the cusp.

□

4.3 Perfections of Igusa curves

In this section we discuss the perfectoid Igusa curves and their Tate parameter spaces. We first recall the perfection functor in characteristic p:

Definition 4.8 ([11], Definition III.2.18)**.** Let $\mathcal{Y}$ be an adic space over $(K^\flat, \mathcal{O}_K^\flat)$. Then there is a perfectoid space $\mathcal{Y}^{\mathrm{perf}}$ over $(K^\flat, \mathcal{O}_K^\flat)$ such that $\mathcal{Y}^{\mathrm{perf}} \sim \varprojlim_F \mathcal{Y}$ where F denotes the relative Frobenius morphism of $\mathcal{Y}$, and where we identify $\mathcal{Y}^{(p)}$ with $\mathcal{Y}$ using that $(K^\flat, \mathcal{O}_K^\flat)$ is perfect. We call $\mathcal{Y}^{\mathrm{perf}}$ the perfection of $\mathcal{Y}$. The formation $\mathcal{Y} \mapsto \mathcal{Y}^{\mathrm{perf}}$ is functorial.

In the case of $\mathcal{Y} = \mathfrak{X}'^*$, we can first take the formal scheme limit $\mathfrak{X}'^{*\mathrm{perf}} = \varprojlim_F \mathfrak{X}'^*(p^{-n}\epsilon)$ in the category of formal schemes. Its generic fibre is then the tilde limit

$$\mathcal{X}'^{*\mathrm{perf}} = \mathfrak{X}'^{*\mathrm{perf}}_\eta \sim \varprojlim_F (\mathfrak{X}'^*(p^{-n}\epsilon))_\eta$$

by Proposition 2.4.2 of [12] and it's easy to check on any affine formal subscheme of $\mathfrak{X}'^*$ that this space is perfectoid. Similarly, we construct the perfection of $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}}$.

Proposition 4.9. *For any cusp c of $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}$, the perfection of the corresponding Tate parameter space $D \hookrightarrow \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)$ fits into a Cartesian diagram*

$$\begin{array}{ccc} D'_\infty & \hookrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} \\ \downarrow & & \downarrow \\ D' & \hookrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon). \end{array}$$

Here the space $D'_\infty := D'^{\mathrm{perf}}$ can be canonically identified with the open subspace of the perfectoid unit disc $\mathrm{Spa}(K^\flat\langle q^{1/p^\infty}\rangle, \mathcal{O}_{K^\flat}\langle q^{1/p^\infty}\rangle)$ defined by $|q| < 1$.

Proof. This follows from [12], Proposition 2.4.3, in the limit over the Cartesian diagrams from Lemma 4.7. □

As before we can cover D'_∞ by affinoid perfectoid subspaces $D'_\infty(|q| \le |\omega^{1/p^k}|)$, and also as before these have canonical models:

Lemma 4.10. *The flat formal scheme* $\mathfrak{D}'_\infty(q/\varpi^{1/p^n}) := \mathrm{Spf}(\mathcal{O}_{K^\flat}\langle (q/\varpi^{1/p^n})^{1/p^\infty}\rangle, (\varpi))$ *is a formal model of* $D'_\infty(|q|^{p^n} \le |\varpi|)$*. The natural inclusion* $\mathcal{O}_{K^\flat}[[q^{1/p^\infty}]] \hookrightarrow \mathcal{O}_{K^\flat}\langle (q/\varpi^{1/p^n})^{1/p^\infty}\rangle$ *induces a morphism of formal schemes*

$$\psi : \mathfrak{D}_\infty(q/\varpi^{1/p^n}) \to \mathrm{Spf}(\mathcal{O}_K[[q^{1/p^\infty}]], (\varpi, q))$$

whose adic generic fibre ψ^{ad}_η *is the inclusion* $D_\infty(|q|^{p^n} \le |\varpi|) \subseteq D_\infty$*. In particular, we have for any cusp of* $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)$ *a canonical formal model* $\mathfrak{D}_\infty(q/\varpi^{1/p^n}) \to \mathfrak{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}}$ *of the restricted Tate parameter space* $D_\infty(q/\varpi^{1/p^n}) \to \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}}$ *at that cusp.*

Proof. Like in Lemma 3.9. □

Lemma 4.11. *The following diagram is Cartesian*

$$\begin{array}{ccc} \mathcal{X}'^*_{\mathrm{Ig}(p^{n+1})}(\epsilon)^{\mathrm{perf}} & \longrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} \\ \downarrow & & \downarrow \\ \mathcal{X}'^*_{\mathrm{Ig}(p^{n+1})}(\epsilon) & \longrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon). \end{array}$$

Proof. The fibre product exists and is perfectoid because the morphism in the bottom horizontal is finite étale. The statement then follows because perfectoid tilde-limits commute with fibre products, and using Lemma 4.7. □

Definition 4.12. As a consequence of the Lemma, we obtain a tower

$$\dots \to \mathcal{X}'^*_{\mathrm{Ig}(p^{n+1})}(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} \to \dots \to \mathcal{X}^*(\epsilon)^{\mathrm{perf}}.$$

of affinoid perfectoid spaces with finite étale transition maps. In particular, the limit of this system exists. We denote it by $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}$ even though so far we have not defined what $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)$ is, but we will later see that this notation is justified.

Proposition 4.13. *Let c be any cusp of* $\mathcal{X}'^*(\epsilon)$*. Then the following diagrams are Cartesian*

$$(i)\quad \begin{array}{ccc} \underline{\mathbb{Z}/p^n\mathbb{Z}} \times D'_\infty & \longrightarrow & D'_\infty \\ \downarrow & & \downarrow f(c) \\ \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} & \longrightarrow & \mathcal{X}^*(\epsilon)^{\mathrm{perf}}. \end{array} \qquad (ii)\quad \begin{array}{ccc} \underline{\mathbb{Z}_p^\times} \times D'_\infty & \longrightarrow & D'_\infty \\ \downarrow & & \downarrow f(c) \\ \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} & \longrightarrow & \mathcal{X}^*(\epsilon)^{\mathrm{perf}}. \end{array}$$

Proof.
1. Follows from Lemma 4.9, Proposition 4.9 and Proposition 4.5 using the Cartesian cube that these three span.
2. Follows in the inverse limit from (i) and Proposition 4.5.2.

□

5 Tilting isomorphisms for modular curves

While so far we have studied modular curves in characteristic 0 and p separately, we now compare the two worlds via tilting. The basis that makes this possible is the following result:

Theorem 5.1 ([11], Corollary III.2.19)**.** *There is a canonical isomorphism*

$$\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a = \mathcal{X}'^*(\epsilon)^{\mathrm{perf}}$$

Let us recall how this is proved: The identification

$$\mathcal{O}_K/p = \mathcal{O}_{K^\flat}/\varpi$$

gives an identification of the reductions $\mathfrak{X}^*/p = \mathfrak{X}'^*/\varpi$ which by an explicit inspection on affine opens extends to a natural isomorphism

$$\mathfrak{X}^*(\epsilon)/p = \mathfrak{X}'^*(\epsilon)/\varpi. \tag{7}$$

Since $\phi : \mathfrak{X}^*(p^{-1}\epsilon) \to \mathfrak{X}^*(\epsilon)$ moreover reduces to Frobenius mod $p^{1-\epsilon}$, we see that when we further reduce equation 7 to $\mathcal{O}_K/p^{1-\epsilon} = \mathcal{O}_{K^\flat}/\varpi^{1-\epsilon}$, then we can also identify the reduction of ϕ with the relative Frobenius morphism. In the inverse limit, this gives the result by [10], Theorem 5.2.

Lemma 5.2. *1. The isomorphism from Theorem 5.1 identifies the cusps of $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ and $\mathcal{X}^*(\epsilon)^{\mathrm{perf}}$.*

2. For any cusp c of $\mathcal{X}^*(\epsilon)$, the tilt of the Tate parameter space $D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ can be identified with the Tate parameter space around the corresponding cusp of $\mathcal{X}^*(\epsilon)^{\mathrm{perf}}$

$$\begin{array}{ccc} D_\infty^\flat & \hookrightarrow & \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a^\flat \\ \| & & \| \\ D'_\infty & \hookrightarrow & \mathcal{X}^*(\epsilon)^{\mathrm{perf}} \end{array}$$

where the identification on the left is induced by the natural isomorphism of closed unit discs $K\langle q^{1/p^\infty}\rangle^\flat = K^\flat\langle q^{1/p^\infty}\rangle$.

Definition 5.3. Let $\mathcal{E} \to \mathcal{X}$ and $\mathcal{E}' \to \mathcal{X}'$ be the analytifications of the respective universal elliptic curves. Since $\mathcal{E}$ is smooth over K, the fibre product $\mathcal{E}_{\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a} := \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a \times_{\mathcal{X}} \mathcal{E}$ exists as a sous-perfectoid adic space. We similarly define $\mathcal{E}'_{\mathcal{X}'(\epsilon)^{\mathrm{perf}}} \to \mathcal{X}'(\epsilon)^{\mathrm{perf}}$.

We denote by $D_n(\mathcal{E}_{\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}) \to \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ the universal anticanonical subgroup of rank n, this is a finite étale morphism of perfectoid space. We moreover denote by $D'_n(\mathcal{E}'_{\mathcal{X}'(\epsilon)^{\mathrm{perf}}}) \to \mathcal{X}'(\epsilon)^{\mathrm{perf}}$ the finite étale perfectoid space given by $\ker V^n$ of $\mathcal{E}'_{\mathcal{X}'(\epsilon)^{\mathrm{perf}}}$.

Lemma 5.4. *The tilt of $D_n(\mathcal{E}_{\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a}) \to \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ is naturally isomorphic to the perfectoid space $D'_n(\mathcal{E}'_{\mathcal{X}'(\epsilon)^{\mathrm{perf}}}) \to \mathcal{X}'(\epsilon)^{\mathrm{perf}}$ over $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a^\flat = \mathcal{X}'(\epsilon)^{\mathrm{perf}}$.*

This Lemma is a slight extension of [11], Lemma III.2.26 from the good reduction locus to the whole uncompactified modular curve (recall that [11] writes $\mathcal{X}$ for the good reduction locus, whereas we use it to denote the whole modular curve).

Proof. It suffices to see this locally on $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$. The case of good reduction is [11] Lemma III.2.26. It therefore suffices to prove the Theorem over the ordinary locus $\mathcal{X}_{\Gamma_0(p^\infty)}(0)$.

Let (R, R^+) be any perfectoid $(K, \mathcal{O}_K)$-algebra and let $\alpha : \mathrm{Spa}(R, R^+) \to \mathcal{X}_{\Gamma_0(p^\infty)}(0)$ be ay morphism, corresponding to an elliptic curve $E|R$ together with the data of subgroups D_n for all R. Since we are over the ordinary locus, we also have canonical subgroups $C_n(E)$ of arbitrary rank and a canonical isomorphism

$$D_n = C_n(E/D_n)^\vee = C_n(E)^\vee$$

constructed as follows: The first isomorphism is induced from the Weil pairing and the short exact sequence

$$0 \to C_n(E/D_n) \to E/D_n[p^n] \to D_n \to 0.$$

The isomorphism $C_n(E/D_n)^\vee = C_n(E)^\vee$ as follows: Since $C_n(E) \cap D_n = 0$, the isogeny $E \to E/D_n$ sends $C_n(E)$ isomorphically onto $C_n(E/D_n)$. Dualizing this isomorphism gives the desired identification $C_n(E/D_n)^\vee = C_n(E)^\vee$.

Let $\alpha^\flat : \mathrm{Spa}(R^\flat, R^{\flat+}) \to \mathcal{X}_{\Gamma_0(p^\infty)}(0)$ be the tilt of α, corresponding to an elliptic curve $E'|R^\flat$ and let $D'_n(E') := \ker V^n(E')$. For any $m \in \mathbb{Z}$ we denote by $E'^{(p^m)}$ the base change along the n-th iterate of absolute Frobenius F^m on R (note that in particular this makes sense for $m < 0$ since R is perfect). Let us write C'_n for $\ker F^n$ on an elliptic curve in characteristic p. Since E' is ordinary, we then have analogous identifications

$$D'_n = C'_n(E'^{(p^{-n})})^\vee = C'_n(E')^\vee$$

where the first isomorphism comes from the short exact sequence

$$0 \to C'_n(E'^{(p^{-n})}) \to E'^{(p^{-n})}[p^n] \xrightarrow{F^n} \ker V^n(E') \to 0$$

and the second comes from identifying kernels of Frobenius under the Verschiebung isogeny $V^n : E' \to E'^{(p^{-n})}$.

We thus see that it suffices to prove that there is a canonical isomorphism of perfectoid spaces $C_n(E)^{\vee\flat} = C'_n(E')^\vee$, functorial in R. To see this, we note that for $C_n(E)$ there is a natural model over R^+ ([1], Proposition 3.2): Indeed, let $E^*_{semi} \to X^*_{\mathcal{O}_K}$ be the semi-abelian scheme extending the universal elliptic curve $E_{\mathcal{O}_K} \to X_{\mathcal{O}_K}$, and let $\mathfrak{E}^*_{semi} \to \mathfrak{X}^*$ be its p-adic completion. Then $\mathfrak{E}^*_{semi}$ has canonical subgroups $\mathfrak{C}_n$ of arbitrary rank over $X_{\mathcal{O}_K}$ which reduce to kernel of Frobenius mod p. Due to the affineness of $\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)$, the map α has a natural formal model $a : \mathrm{Spf}(R^+) \to \mathfrak{X}^*_{\Gamma_0(p^\infty)}(0) \to \mathfrak{X}^*$ and by pulling back $\mathfrak{C}_n$ we thus obtain a canonical model $\mathfrak{C}_n(E)$ of $C_n(E)$ over R^+ with étale dual.

A similar argument using the semi-abelian scheme $\mathfrak{E}'^*_{semi} \to \mathfrak{X}'^*$ gives a canonical formal model $\mathfrak{C}'_n(E')$ of $C_n(E')$ over $\mathrm{Spf}(R^{\flat+})$ with étale dual. The natural isomorphism

$$\mathfrak{E}^*_{semi}/p = \mathfrak{E}'^*_{semi}/\varpi$$

then induces an isomorphism $\mathfrak{C}_n(E)/p = \mathfrak{C}'_n(E')/\varpi$ which is functorial in R and α since both spaces were defined by pullback from the universal situation. Via Cartier duality we now obtain a natural isomorphism

$$\mathfrak{C}_n(E)^\vee/p = \mathfrak{C}'_n(E')^\vee/\varpi.$$

The tilting equivalence ([10], Theorem 5.2) now implies that $C_n(E)^{\vee\flat} = C'_n(E')^\vee$. $\square$

The following Lemma gives a more explicit description of this isomorphism over the Tate parameter spaces:

Lemma 5.5. *Let c be any cusp of $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$. Then over the corresponding Tate parameter space $\mathring{D}_\infty \to \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$, the restriction of $D_n(\mathcal{E}_{\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a})$ is canonically isomorphic to $\underline{\mathbb{Z}/p^n\mathbb{Z}}$ via the generator q^{1/p^n} of $\langle q^{1/p^n}\rangle \subseteq \mathrm{T}(q)[p^n]$. Similarly we have $D'_n(\mathcal{E}_{\mathcal{X}'(\epsilon)^{\mathrm{perf}}}) = \underline{\mathbb{Z}/p^n\mathbb{Z}}$ on $\mathring{D}'_\infty \to \mathcal{X}'(\epsilon)^{\mathrm{perf}}$. The isomorphism $D_n^\flat = D'_n$ over $\mathring{D}'$ form Lemma 5.4 is then the one that commutes with the canonical isomorphism $\underline{\mathbb{Z}/p^{\mathrm{n}}\mathbb{Z}}^\flat = \underline{\mathbb{Z}/p^n\mathbb{Z}}$.*

Proof. In the proof of the Lemma, the isomorphism $D_n^\flat = D'_n$ was constructed using the canonical identification of the canonical subgroups on the special fibre. Since $\mathring{D} \hookrightarrow \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ arises as the restriction of the generic fibre of

$$\mathrm{Spf}(\mathcal{O}_{K^\flat}[[q]]) \to \mathfrak{X}^*_{\Gamma_0(p^\infty)}(0)_a,$$

to the open subspace away from the cusp, this reduces to a statement about the pullback of the semiabelian scheme to $\mathrm{Spf}(\mathcal{O}_K[[q]])$. But since this is the semiabelian scheme associated to the Tate curve, it's canonical subgroup is canonically isomorphic to μ_{p^n} over $\mathcal{O}_K[[q]]$ (cite somewhere in [3]), and similarly over $\mathcal{O}_K[[q]]$. On Cartier duals, the canonical isomorphism $\mu_{p^n} \to C_n(\mathrm{T}(q))$ gives the desired trivialisation $D_n(\mathrm{T}(q)) \to \underline{\mathbb{Z}/p^n\mathbb{Z}}$ on $\mathring{D}_\infty$. Similarly for $D'_n(\mathrm{T}(q)) \to \underline{\mathbb{Z}/p^n\mathbb{Z}}$ on $\mathring{D}'_\infty$.

The Lemma follows since the isomorphisms $\mu_{p^n} \to C_n(\mathrm{T}(q))$ over $\mathcal{O}_K[[q]]$ and $\mu_{p^n} \to C'_n(\mathrm{T}(q))$ over $\mathcal{O}_{K^\flat}[[q]]$ are identified upon reduction to $\mathcal{O}_K/p[[q]] = \mathcal{O}_{K^\flat}/\varpi[[q]]$. □

Theorem 5.6. *1. For any $n \in \mathbb{N}$, there is a canonical isomorphism*

$$\begin{array}{ccc} \mathcal{X}^*_{\Gamma_1(p^n)\cap\Gamma_0(p^\infty)}(\epsilon)^\flat_a & \xrightarrow{\sim} & \mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} \\ \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a & = & \mathcal{X}'^*(\epsilon)^{\mathrm{perf}} \end{array}$$

which is $(\mathbb{Z}/p^n\mathbb{Z})^\times$-equivariant and makes the diagram commute.

2. In the limit, we obtain a canonical isomorphism

$$\begin{array}{ccc} \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a & \xrightarrow{\sim} & \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \\ \downarrow & & \downarrow \\ \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a & = & \mathcal{X}'^*(\epsilon)^{\mathrm{perf}} \end{array}$$

which is $\mathbb{Z}_p^\times$-equivariant and makes the diagram commute.

3. Over any cusp of $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$, the tilt of the cusp morphism $\underline{\mathbb{Z}_p^\times} \times D_\infty \hookrightarrow \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ fits into the commutative diagram

$$\begin{array}{ccc} \underline{\mathbb{Z}_p^\times} \times D^\flat_\infty & \hookrightarrow & \mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)^\flat_a \\ \| & & \| \\ \underline{\mathbb{Z}_p^\times} \times D'^{\mathrm{perf}} & \hookrightarrow & \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}}. \end{array}$$

Proof. For any $n \in \mathbb{N}$, the functorial isomorphism from Lemma 5.4 induces by the moduli interpretations an isomorphism away from the cusps:

$$\begin{array}{ccc} \mathcal{X}_{\Gamma_1(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)^\flat_a & = & \mathcal{X}'_{\mathrm{Ig}(p^m)}(\epsilon)^{\mathrm{perf}} \\ \downarrow & & \downarrow \\ \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a & = & \mathcal{X}'(\epsilon)^{\mathrm{perf}} \end{array}$$

In order to prove part 1 of the Theorem, we need to show that this extends over the cusps. To this end, fix a cusp c of $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$. Recall from Lemma 5.2 that on the Tate parameter spaces at c, the isomorphism $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)^\flat_a \to \mathcal{X}'^*(\epsilon)^{\mathrm{perf}}$ restricts to the canonical isomorphism $D^\flat_\infty = D'_\infty$. Using the description of the Tate parameters in $\mathcal{X}^*_{\Gamma_1(p^m)\cap\Gamma_0(p^\infty)}(\epsilon)_a \to \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$ by Proposition 3.11 and similarly in $\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} \to \mathcal{X}^*(\epsilon)^{\mathrm{perf}}$ by Proposition 4.13, the description of the isomorphism $D^\flat_n = \mathring{D}'_n$ on Tate parameter in Lemma 5.5 now shows that the above diagram restricts over $f(c) : \mathring{D}_\infty \hookrightarrow \mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$ to the diagram

$$\begin{array}{ccc} \underline{\mathbb{Z}/p^n\mathbb{Z}} \times \mathring{D}^\flat_\infty & = & \underline{\mathbb{Z}/p^n\mathbb{Z}} \times \mathring{D}'_\infty \\ \downarrow & & \downarrow \\ \mathring{D}^\flat_\infty & = & \mathring{D}'_\infty \end{array}$$

where the morphism on the top line is the one which is the identity on $\underline{\mathbb{Z}/p^n\mathbb{Z}}$. It is clear from this description that the isomorphism extends uniquely to an isomorphism $\underline{\mathbb{Z}/p^n\mathbb{Z}} \times D^\flat_\infty \to \underline{\mathbb{Z}/p^n\mathbb{Z}} \times D'_\infty$. But this means that the first diagram extends over the cusp c. This proves the first part of the Theorem.

Part 2 is an immediate consequence in the limit $n \to \infty$.

Part 3 follows from the second diagram together with Theorem 3.1 and Proposition 4.13 which describe the maps between Tate parameter spaces in the two towers for $n \to \infty$. □

6 q-expansion principles

In this section, we prove various perfectoid q-expansion principles for function on the spaces $\mathcal{X}_{\Gamma_0(p^\infty)}(\epsilon)_a$, $\mathcal{X}_{\Gamma_1(p^\infty)}(\epsilon)_a$ and $\mathcal{X}_{\Gamma(p^\infty)}(\epsilon)_a$.

6.1 Detecting vanishing

The classical q-expansion principle, §1 [8], states that a modular form vanishes if and only if its q-expansions do. The perfectoid q-expansion principle says that a function on one of the infinite level spaces vanishes if its q-expansions do. More precisely, we prove:

Proposition 6.1 (q-expansion principle I). *Let $c_0, \dots, c_m$ be a collection of cusps of $\mathcal{X}'^*$ such that each connected component of $\mathcal{X}^*$ contains at least one c_i. Let $\sqcup_{i=1}^m D \hookrightarrow X^*$ be the corresponding Tate parameter spaces; Let $n \in \mathbb{N} \cup \{\infty\}$, let Γ be one of $\Gamma_0(p^n), \Gamma_1(p^n), \Gamma(p^n)$ and let $\mathcal{D} \to \mathcal{X}^*_\Gamma(\epsilon)_a$ be the pullback. Then the map $\mathcal{O}(\mathcal{X}^*_\Gamma(\epsilon)_a) \to \mathcal{O}(D)$ is injective.*

Corollary 6.2. *Let $c_0, \dots, c_m$ be a collection of cusps of $\mathcal{X}'^*$ such that each connected component of $\mathcal{X}^*$ contains at least one c_i. Then restriction of functions gives injective maps*

$$\mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a) \hookrightarrow \prod_{i=1}^m \mathcal{O}_K[[q^{1/p^\infty}]][1/p]$$

$$\mathcal{O}(\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a) \hookrightarrow \prod_{i=1}^m \mathrm{Map}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathcal{O}_K[[q^{1/p^\infty}]])[1/p]$$

$$\mathcal{O}(\mathcal{X}^*_{\Gamma(p^\infty)}(\epsilon)_a) \hookrightarrow \prod_{i=1}^m \mathrm{Map}_{\mathrm{cts}}(K_0(p^\infty), \mathcal{O}_K[[q^{1/p^\infty}]])[1/p].$$

This is basically an analogue of saying that for any affine irreducible integral variety over K, completion at any K-point gives rise to an injection on function, which is a consequence of Krull's Intersection Theorem. The perfectoid situation is a bit more subtle, since Krull's Intersection Theorem requires Noetherianess. In the above case, one can descent to the Noetherian situation using that modular curves are already defined over $\mathbb{Z}_p$.

Our proof of the Proposition is in two steps: We first consider $\mathcal{X}^*_{\Gamma_0(p^\infty)}(0)_a$ where it is easy to reduce to the Noetherian case. In a second step, we then show that restriction of functions from $\mathcal{X}^*_{\Gamma_1(p^\infty)}(\epsilon)_a$ to $\mathcal{X}^*_{\Gamma_1(p^\infty)}(0)_a$ is injective, which is a straight-forward computation on power series. We start with the case $\epsilon = 0$. For this we need the following Lemma:

Lemma 6.3. *Let R be any ring. Let $\varpi \in R$ be a non-zero-divisor. Let $\varphi : A \to B$ be a morphism of R-algebras. Assume that B is a flat R-algebra and that $\varphi : A/\varpi \hookrightarrow B/\varpi$ is injective. Then the induced map on ϖ-adic completions $\hat{\varphi} : \hat{A} \hookrightarrow \hat{B}$ is injective.*

Proof. Since ϖ is a non-zero-divisor, for any n the following sequence is short exact:

$$0 \to R/\varpi \xrightarrow{\cdot\varpi^{n-1}} R/\varpi^n \to R/\varpi^{n-1} \to 0$$

After tensoring with $A \to B$, the five-lemma applies (1using that B is flat over R) and shows inductively that $A/\varpi^n \to B/\varpi^n$ is injective for all n. The Lemma follows in the limit. □

Proposition 6.4. *Let $c_0, \dots, c_m$ be a collection of cusps of X^* such that each connected component of X^* contains at least one c_i. Let $\sqcup_{i=1}^m D \hookrightarrow \mathcal{X}^*$ and $\sqcup_{i=1}^m \operatorname{Spf} \mathcal{O}_{K^\flat}[[q]] \hookrightarrow \mathfrak{X}^*$ be the corresponding Tate parameter space.*

Let Γ by any of the level structures $\Gamma = \Gamma_0(p^n), \Gamma_1(p^n), \Gamma(p^n)$ for any $n \in \mathbb{N}_0 \cup \{\infty\}$ and let $\mathcal{Y} = \mathcal{X}^*_\Gamma(0)_a$. Let $t : \mathcal{D} \to \mathcal{Y}$ be the pullback of the Tate parameter space $\sqcup_{i=1}^m D \hookrightarrow \mathcal{X}^*$. to $\mathcal{Y}$. Then the map of sections $\mathcal{O}(\mathcal{Y}) \hookrightarrow \mathcal{O}(\mathcal{D})$ is injective.

Proof. It therefore suffices to consider the cases of $\Gamma = \Gamma_0$ and $\Gamma = \Gamma_1$: The case of $\Gamma = \Gamma(p^n)$ then follows from Lemma 3.19.

Let therefore $\Gamma = \Gamma_0(p^n)$ or $\Gamma_1(p^n)$. We first we note that the space $\mathcal{Y}$ has a canonical formal model $\mathfrak{Y} = \mathfrak{X}^*_\Gamma(0)_a$, which is affine, say $\mathfrak{Y} = \operatorname{Spf}(R)$. We first consider the case of $n < \infty$. Let $C_1, \dots, C_l$ be the cusps of $\mathfrak{Y}^*$ lying over the cusps $c_1, \dots, c_m$ of X^* and let $\sqcup_{i=1}^l \operatorname{Spf} \mathcal{O}_K[[q]] \to \mathfrak{Y}$ be the completion along the subscheme of cusps $C_1, \dots, C_l$. It suffices to show that the map on global sections $\varphi : R \to \prod_{i=1}^m \mathcal{O}_{K^\flat}[[q]]$ is injective: Indeed, it then follows that the induced map $R \to \prod_{i=1}^m \mathcal{O}_K\langle q/\varpi^k\rangle$ for any $k \in \mathbb{N}$ is injective, and after tensoring with K, in the direct limit over k these glue to the morphism $\mathcal{O}(\mathcal{D}) \hookrightarrow \mathcal{O}(\mathcal{Y})$.

Lemma 6.3 further reduces us to proving that the reduction $R/p \to \prod_{i=1}^m \mathcal{O}_K/p[[q]]$ is injective. This reduction can be interpreted as follows: Let $Y = X^*_{\mathcal{O}_K/p,ord}$ in the case of $\Gamma = \Gamma_0(p^n)$, and $Y = \mathrm{Ig}_{\mathcal{O}_K/p,p^n,ord}$ in the case of $\Gamma = \Gamma_1(p^n)$ and write $Y = \operatorname{Spec}(A)$. Then the reduction of φ can be identified with the morphism $A \to \prod \mathcal{O}_K/p^n[[q]]$ given by the global section of the completion $\sqcup \operatorname{Spf}(\mathcal{O}_K/p^n[[q]]) \to Y$ at the cusps $C_1, \dots, C_l$. Since $\mathcal{O}_K/p$ is a flat $\mathbb{F}_q = \mathbb{F}_p[\zeta_N]$-algebra, it now suffices to prove that for $Y_{\mathbb{F}_q} = \operatorname{Spec}(A_{\mathbb{F}_q})$ any one of schemes $X'^*_{\mathbb{F}_q,ord}$, $\mathrm{Ig}_{\mathbb{F}_q,p^n,ord}$, completion at the cusps gives an injection $A_{\mathbb{F}_q} \to \prod_{i=1}^l \mathbb{F}_q[[q]]$.

We are now in the situation of smooth varieties over a field, and by considering each connected component separately, it suffices to prove that for an integral Noetherian ring A, completion at any maximal ideal $\mathfrak{m} \subseteq A$ gives an injection $A \to \hat{A}_\mathfrak{m}$. But this follows from Krull's intersection theorem, which says that $A_\mathfrak{m} \hookrightarrow \hat{A}_\mathfrak{m}$ is injective.

This finishes the prove in the case of $n < \infty$. The case of $n = \infty$ can be deduced in the limit: For any $m \in \mathbb{N}$ let $\mathfrak{Y}_m = \mathfrak{X}^*_{\Gamma_0(p^m)}(0)$ or $\mathfrak{Y}_m = \mathfrak{X}^*_{\Gamma_1(p^m)}(0)$. We first set up some more technical notation: For each $m < \infty$, the morphism $\mathcal{O}(\mathfrak{Y}_m) \to \prod \mathcal{O}_K[[q]]$ is not only continuous for the (p, q)-adic topology on the target, but also for the p-adic topology, as can be seen by taking the limit of the reduction mod p^r. We obtain a corresponding morphism of formal schemes $\sqcup \operatorname{Spf}(\mathcal{O}_K[[q]], (p)) \to \mathcal{Y}_m$ where the notation on the left hand side emphasizes that we now consider the p-adic topology. Composing with the morphism $\sqcup \operatorname{Spf}(\mathcal{O}_K\langle q/\omega^{1/p^k}\rangle) \to \operatorname{Spf}(\mathcal{O}_K[[q]], (p))$ for any $k \in \mathbb{N}$ gives on the generic fibre the affinoid restricted Tate parameter spaces $\sqcup D(|q| \leq |\omega^{1/p^k}\rangle|) \to \mathcal{X}^*_\Gamma(0)$ from before.

Passing to $n = \infty$, consider for any $m \in \mathbb{N} \cup \{\infty\}$ the pullback of p-adic formal schemes

$$\begin{array}{ccc} \mathfrak{D}_m & \longrightarrow & \mathfrak{Y}_m \\ \downarrow & & \downarrow \\ \sqcup \operatorname{Spf}(\mathcal{O}_K[[q]], (p)) & \longrightarrow & \mathfrak{X}^*. \end{array}$$

By Lemma 3.10, the same argument as before, namely covering D_∞ by the affinoid spaces $D_\infty(|q| \leq |\omega^{1/p^k}|)$, shows that it suffices to prove that $\mathcal{O}(\mathfrak{Y}_\infty) \to \mathcal{O}(\mathfrak{D}_\infty)$ is injective. By Lemma 6.3, it suffices to prove this for the reduction mod p. But since $\mathfrak{D}_\infty \to \mathfrak{Y}_m$ is the projective limit of the $\mathfrak{D}_m \to \mathfrak{Y}_m$, this follows in the direct limit from the fact that $\mathfrak{O}(\mathfrak{Y}_m)/p \to \mathfrak{O}(\mathfrak{D}_m)/p$ is injective, as we have already seen. □

The proof of Proposition 6.1 is completed by the following Proposition, which is also relevant on its own for applications to modular forms:

Lemma 6.5. *Let $\mathcal{Y} \to \mathcal{X}^*$ be one of the following adic spaces: $\mathcal{X}^*_{\Gamma_0(p^n)}$, $\mathcal{X}_{\Gamma_0(p^n)}$, $\mathcal{X}^*_{\Gamma_1(p^n)}$, $\mathcal{X}_{\Gamma_1(p^n)}$, $\mathcal{X}^*_{\Gamma(p^n)}$, $\mathcal{X}_{\Gamma(p^n)}$, each for any $n \in \mathbb{N}_0 \cup \{\infty\}$. Then the open immersion $\mathcal{Y}(0) \to \mathcal{Y}(\epsilon)$ on global section gives an injection $\mathcal{O}(\mathcal{Y}(\epsilon)) \to \mathcal{O}(\mathcal{Y}(0))$.*

The proof boils down to the following Lemma:

Lemma 6.6. *Let A be any ring, let $0 \neq \varpi \in A$ be a non-zero-divisor and let $H \in A$ be such that its image in A/ϖ is a non-zero-divisor. Endow A with the ϖ-adic topology. Then*

$$\varphi : A\langle X\rangle/(XH - \varpi) \xrightarrow{X\mapsto\varpi X} A\langle X\rangle/(XH - 1)$$

is an injective morphism. Similarly for $\varphi : A[X]\rangle/(XH - \varpi) \xrightarrow{X\mapsto\varpi X} A[X]\rangle/(XH - 1)$.

Proof. We first note that the assumption on $H \in A$ implies that H is a non-zero-divisor in any A/ϖ^n: This follows from Lemma 6.3 applied to the morphism of A-algebras $A \xrightarrow{\cdot H} A$.

Suppose $f = \sum a_n X^n$ is in the kernel of $A\langle X\rangle \to A\langle X\rangle/(XH - \varpi) \xrightarrow{\varphi} A\langle X\rangle/(XH - 1)$. Then there is $g = \sum b_n X^n$ such that

$$f(\varpi X) = \sum a_n \varpi^n X^n = (XH - 1)g = (XH - 1)\sum b_n X^n.$$

Reducing mod ϖ^m, we see that

$$a_0 + \cdots + a_{m-1}\varpi^{m-1}X^{m-1} \equiv (XH - 1)\sum b_n X^n \bmod \varpi^m$$

By comparing leading coefficients in $A/\varpi^m[X]$, and since H is not a zero-divisor mod ϖ^m, we conclude that $b_k \equiv 0 \bmod \varpi^m$ for $k \geq m - 1$.

Consequently, there are elements $c_m = b_m/\varpi^{m+1} \in A$ for all m and we have in $A[[X]]$

$$f' := (XH - \varpi)\sum \frac{b_m}{\varpi^{m+1}} X^m \overset{X\mapsto\varpi X}{\longmapsto} (XH - 1)\sum b_m X^m.$$

Thus $f'(\varpi X) = f(\varpi X)$ in $A[[X]]$ which implies $f' = f$ since ϖ is not a zero divisor.

It remains to prove that $\sum c_m X^m$ converges in $A\langle X\rangle$. To see this, we use that $f \in A\langle X\rangle$. For every $k \in \mathbb{N}$ there is N_k such that $v(a_m) \geq k$ for all $m \geq N_k$. In particular, we then have $v(\varpi^m a_m) \geq k + m$ for all $m \geq N_k$. Consequently, for all $m \geq N_k$

$$a_0 + \cdots + a_{m-1}\varpi^{m-1}X^{m-1} \equiv (XH - 1)\sum b_m X^m \bmod \varpi^{m+k}.$$

This shows that $v(b_{m-1}) \geq m + k$, and thus $v(c_m) \geq k$ for all $m \geq N_k$. Thus $\sum c_m X^m \in A\langle X\rangle$ as desired.

We conclude that f is already in $(XH - \varpi)A\langle X\rangle$. Thus φ is injective. □

proof of Proposition 6.5. We prove instead: Let $\mathfrak{U} \to \mathfrak{X}$ be an open in one of the following formal schemes: $\mathfrak{X}_{\Gamma_0(p^n)}$, $\mathfrak{X}_{\Gamma_1(p^n)}$, $\mathfrak{X}_{\Gamma(p^n)}$, each for any $n \in \mathbb{N}_0 \cup \{\infty\}$. Then the natural map $\mathfrak{Y}(0) \to \mathfrak{Y}(\epsilon)$ induces an injection

$$\mathcal{O}(\mathfrak{Y}(\epsilon)|_{\mathfrak{U}}) \to \mathcal{O}(\mathfrak{Y}(0)|_{\mathfrak{U}}). \tag{8}$$

The statement is local, so we may without loss of generality assume that $\mathfrak{U}$ is an affine open on which ω is trivial. Then $\mathfrak{Y}(\epsilon)|_{\mathfrak{U}} = \mathrm{Spf}(R\langle X\rangle/(X\,\mathrm{Ha}-p^\epsilon))$ and $\mathfrak{Y}(0)|_{\mathfrak{U}} = \mathrm{Spf}(R\langle X\rangle/(X\,\mathrm{Ha}-1))$ and the map $\mathfrak{Y}(\epsilon)|_{\mathfrak{U}} \to \mathfrak{Y}(0)|_{\mathfrak{U}}$ is given on functions by

$$\varphi : R\langle X\rangle/(X\,\mathrm{Ha}-p^\epsilon) \xrightarrow{X\mapsto p^\epsilon X} R\langle X\rangle/(X\,\mathrm{Ha}-1).$$

We would like to apply Lemma 6.6 with $\varpi = p^\epsilon$. This requires that ϖ is a non-zero-divisor, which is clear since $\mathfrak{U}$ is flat over $\mathrm{Spf}(\mathcal{O}_K)$, and that Ha is a non-zero divisor in R/p^ϵ.

To see the latter, we may without loss of generality assume that the function Ha on the open $\mathfrak{U} \subseteq \mathfrak{Y}$ already arises by base change from an irreducible affine open subscheme of $X_{\mathbb{F}_q}$. But then Ha is the flat base change of a non-zero function on an integral affine scheme, and thus is a non-zero divisor.

Thus Lemma 6.6 applies, which shows that the morphism (8) is injective. The Proposition in the case of $\mathcal{Y} = \mathcal{X}_{\Gamma_0(p^n)}, \mathcal{X}_{\Gamma_1(p^n)}, \mathcal{X}_{\Gamma(p^n)}$ follows from the first part because on any affine open formal subschemes $\mathrm{Spf}(R) \subseteq \mathfrak{Y}$, the functions on the adic generic fibre are given by tensoring with $\otimes_{\mathcal{O}_K} K$. The cases of the spaces $\mathcal{Y} = \mathcal{X}^*_{\Gamma_0(p^n)}, \mathcal{X}^*_{\Gamma_1(p^n)}, \mathcal{X}_{\Gamma(p^n)}$ by using the open cover $\mathcal{X}^* = \mathcal{X}^*(0) \cup \mathcal{X}$ since the boundary is disjoint of the supersingular locus. □

This finishes the proof of Proposition **??** □

6.2 Tate traces and detecting the level

While the transition from $\Gamma_0(p^\infty)$ to $\Gamma(p^\infty)$ is controlled by the Galois action, the transition of $\Gamma_0(p)$ to $\Gamma_0(p^\infty)$ can be controlled by normalised Tate traces, as discussed in [11], III.2.4 and [1], Proposition 6.2. We recall:

Proposition 6.7 ([11], Corollary III.2.23)**.** *The normalised Tate traces*

$$\mathrm{tr}_{n,m} : \mathcal{O}_{\mathfrak{X}^*(p^{-n}\epsilon)} \to \mathcal{O}_{\mathfrak{X}^*(p^{-m}\epsilon)}[1/p]$$

of the morphism $\phi : \mathfrak{X}^*(p^{-n}\epsilon) \to \mathfrak{X}^*(p^{-m}\epsilon)$ *for* $0 \le m \le n \in \mathbb{N}$ *in the limit* $n \to \infty$ *give rise to compatible continuous morphisms*

$$\mathrm{tr}_m : \mathcal{O}_{\mathfrak{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a} \to \mathcal{O}_{\mathfrak{X}^*(p^{-m}\epsilon)_a}[1/p].$$

Proof. This is [11], Corollary III.2.23, except that we use $\mathfrak{X}^*$ instead of $\mathfrak{X}$: This is possible since in contrast to the higher genus Siegel moduli spaces, the minimal compactification of the modular curve $\mathfrak{X}^*$ is a smooth formal scheme, and thus Corollary III.2.22 applies over all of $\mathfrak{X}^*$, not just over $\mathfrak{X}$, which means that the proof of III.2.23 goes through for $\mathfrak{X}^*$. □

Definition 6.8. On the generic fibre, the trace tr_m for $m = 0$ extends to a K-linear Tate trace map of sheaves on $\mathfrak{X}^*(\epsilon)$ that we denote by

$$\mathrm{tr} : \mathcal{O}_{\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a} \to \mathcal{O}_{\mathcal{X}^*(\epsilon)}$$

(here the sheaves are tacitly pushed forward to $\mathfrak{X}^*_{\Gamma_0(p)}$ along the maps of locally ringed spaces $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a \to \mathfrak{X}^*_{\Gamma_0(p)}$ and $\mathcal{X}^*_{\Gamma_0(p)} \to \mathfrak{X}^*_{\Gamma_0(p)}$, but we omit this from notation).

Over the Tate parameter spaces, the normalised trace can be described as follows:

Proposition 6.9. *Let c be any cusp of $\mathcal{X}^*(\epsilon)$, with corresponding Tate parameter space $D \hookrightarrow \mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)$. Then the normalised Tate trace fits into a commutative diagram*

$$\begin{array}{ccccc} \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a) & \longrightarrow & \mathcal{O}(D_\infty) & & \sum_{m\in\mathbb{Z}[1/p]_{\geq 0}} a_m q^m \\ \downarrow \mathrm{tr} & & \downarrow \mathrm{tr} & & \downarrow \\ \mathcal{O}(\mathcal{X}^*(\epsilon)) & \longrightarrow & \mathcal{O}(D) & & \sum_{m\in\mathbb{Z}_{\geq 0}} a_n q^n \end{array}$$

where the map on the right is given by forgetting all coefficients a_m for $m \notin \mathbb{N}$.

Proof. By continuity, the trace morphism is uniquely determined by the finite level normalised traces $\mathrm{tr}_{n,0} : \mathcal{O}_{\mathfrak{X}^*(p^{-n}\epsilon)} \to \mathcal{O}_{\mathfrak{X}^*(\epsilon)}[1/p]$. It thus suffices to determine the effect on q-expansions at finite level: By Lemma 2.10, this is the trace of the morphism $\mathcal{O}_K[[q]] \to \mathcal{O}_K[[q]]$, $q \mapsto q^{p^n}$, or equivalently $\mathcal{O}_K[[q]] \to \mathcal{O}_K[[q^{1/p^n}]]$, $q \mapsto q$. Since the extension of fraction fields is Galois with Galois automorphisms $q^{1/p^n} \mapsto q^{1/p^n}\zeta_{p^n}^d$ for $d \in \mathbb{Z}/p^n\mathbb{Z}$, this trace is

$$\sum_{k=0}^{\infty} a_{\frac{k}{p^n}} q^{\frac{k}{p^n}} \mapsto \frac{1}{p^n}\sum_{k=0}^{\infty} a_{\frac{k}{p^n}}(1+\zeta_{p^n}^k+\dots+\zeta_{p^n}^{dk})q^{\frac{k}{p^n}} = \sum_{m=1}^{\infty} a_m q^m$$

since $1+\zeta_{p^n}^k+\dots+\zeta_{p^n}^{dk} = 0$ unless $p^n|k$. This gives the desired description. □

Corollary 6.10 (q-expansion principle II)**.** *Let $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a)$ on $\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a$. Then for any $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the following are equivalent:*

1. *f is already a function on $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$.*
2. *The q-expansion of f at every cusp is already in $\mathcal{O}_K[[q^{1/p^n}]][1/p] \subseteq \mathcal{O}_K[[q^{1/p^\infty}]][1/p]$.*
3. *On every connected component of $\mathcal{X}^*_{\Gamma_0(p^n)}(\epsilon)_a$ there is at least one cusp at which the q-expansion of f is already in $\mathcal{O}_K[[q^{1/p^n}]][1/p] \subseteq \mathcal{O}_K[[q^{1/p^\infty}]][1/p]$.*

Proof. It suffices to prove that 3 implies 1. A function $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma_0(p^\infty)}(\epsilon)_a)$ is already in $\mathcal{O}(\mathcal{X}^*(\epsilon))$ if and only if $\mathrm{tr}(f) = f$. By Theorem **??**, this can be checked on sufficiently many q-expansions. By Proposition 6.9, at any given cusp we have $\mathrm{tr}(f) = f$ if and only if the q-expansion at that cusp is already in $\mathcal{O}_K[[q]][1/p]$. □

We also have the following analogue in characteristic p:

Corollary 6.11 (q-expansion principle II, characteristic p version)**.** *Let $f \in \mathcal{O}(\mathcal{X}^*(\epsilon)^{\mathrm{perf}})$. Then the following are equivalent:*

1. *f is already a function on $\mathcal{X}'^*(\epsilon)$.*
2. *The q-expansion of f at every cusp is already in $\mathcal{O}_{K^\flat}[[q]][1/\varpi] \subseteq \mathcal{O}_{K^\flat}[[q^{1/p^\infty}]][1/\varpi]$.*
3. *On every connected component of $\mathcal{X}'^*(\epsilon)$ there is at least one cusp at which the q-expansion of f is already in $\mathcal{O}_{K^\flat}[[q]][1/\varpi] \subseteq \mathcal{O}_{K^\flat}[[q^{1/p^\infty}]][1/\varpi]$.*

The analogous statement for $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)^{\mathrm{perf}} \to \mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(\epsilon)$ is also true.

Proof. Like the proof of the last Corollary, using Proposition **??**, Proposition **??** and Lemma **??** □

6.3 Detecting boundedness

Proposition 6.12 (q-expansion principle III)**.** *A function $f \in \mathcal{O}(\mathcal{X}^*_{\Gamma_1(p^\infty)}(0)) = S$ is contained in $S^\circ = \mathcal{O}^+(\mathcal{X}^*_{\Gamma_1(p^\infty)}(0))$ if and only if its q-expansion is in $\mathcal{O}_K[[q^{1/p^\infty}]]$. In particular, the natural morphism*

$$\varphi : S^\circ/p \to \mathrm{Map}_{lc}(\mathbb{Z}_p, \mathcal{O}_K/p[[q^{1/p^\infty}]])$$

is injective. The analogous statement for $\mathcal{X}'^*_{Ig(p^\infty)}(0)^{\mathrm{perf}} = \mathrm{Spa}(S^\flat, S^{\flat\circ})$ is also true: An element of $S^\flat$ is in $S^{\flat\circ}$ if and only if its q-expansion is in $\mathcal{O}_{K^\flat}[[q^{1/p^\infty}]]$.

Note: Should replace proof by part of the proof of Proposition 9.32

Proof. We will prove that φ is injective, this implies all other statements, which can be seen as follows: Let $f \in S$ be any element with q-expansion in $\mathcal{O}_K/p[[q^{1/p^\infty}]]$. Since S is Tate, there is an integer $n \geq 1$ such that $p^n f \in S^\circ$. Assume that n is minimal. Then since $n \geq 1$, the q-expansion of $p^n f$ is in $\mathrm{Map}_{cts}(\mathbb{Z}_p, p\mathcal{O}_K[[q^{1/p^\infty}]])$. Since φ is injective, this implies $p^n f \in pS^\circ$, thus $p^{n-1} f \in S^\circ$. The only way this doesn't contradict $n \geq 1$ being minimal is that $n = 1$, hence $f \in S^\circ$.

In fact, it suffices to prove that the kernel of φ is almost zero: If this is the case, then any $f \in S^\circ$ that lands in the kernel is such that for any $\varpi \in \mathcal{O}_K$ with $0 < |\varpi| < 1$ one has $\varpi f \in pS^\circ$, thus $|f| \leq |\varpi|^{-1}|p|$, then we must have $|f| \leq |p|$ and thus $f \in pS^\circ$.

It just suffices to prove that φ is almost injective. To see this, we first consider the case of the tame level modular curve over $\mathbb{Z}_p$ and its ordinary locus $\mathfrak{X}^*_{\mathbb{Z}_p}(0)$. Recall that this is smooth, since its open in $\mathfrak{X}^*_{\mathbb{Z}_p}$, and in particular normal. It is moreover affine, say $\mathfrak{X}^*_{\mathbb{Z}_p}(0) = \mathrm{Spf}(R)$ and as a consequence of the normality we have $\mathcal{X}^*_{\mathbb{Z}_p}(0) = \mathrm{Spa}(R[1/p], R)$, ie R is the ring of bounded functions on $\mathcal{X}^*_{\mathbb{Z}_p}(0)$. The classical q-expansion principle, more precisely Proposition 2.7.1 in [8] then guarantees that

$$R^\circ/p \to \mathcal{O}_K/p[[q]] \tag{9}$$

is injective (the way this is proved is that one multiplies f mod p with powers of E_{p-1} until the function extends to the whole modular curve, ie is a classical modular form, without changing the q-expansion mod p. Then the classical q-expansion principle applies).

We now switch to the situation over $K^\flat$ and consider the tame curve $\mathfrak{X}'^*(0)$ with its perfection $\mathfrak{X}'^*(0)^{\mathrm{perf}}$. Since $\mathfrak{X}'^*(0) = \mathrm{Spf}(S_0)$ is affine, we have $\mathfrak{X}'^*(0)^{\mathrm{perf}} = \mathrm{Spf}(S_0^{\mathrm{perf}})$ where S_0^{perf} is the completion of $\varinjlim_F S_0$. Write $\mathcal{X}'^*(0)^{\mathrm{perf}} = \mathrm{Spa}(A', A'^\circ)$, then $S_0 \overset{a}{=} A'^\circ$. By taking the direct limit over relative Frobenius of 9, we thus get an almost injection

$$A'^\circ/\varpi \to \mathcal{O}_K/\varpi[[q^{1/p^\infty}]],$$

meaning that the kernel is almost zero. It remains to base change from $\mathcal{X}'^*(0)^{\mathrm{perf}}$ to $\mathcal{X}'^*_{\mathrm{Ig}(p^\infty)}(0)^{\mathrm{perf}} = \mathrm{Spa}(A'_{Ig(p^n)}, A'^+_{Ig(p^n)})$. But since this is a tilde-limit of the finite étale morphisms

$$\mathcal{X}'^*_{\mathrm{Ig}(p^n)}(\epsilon)^{\mathrm{perf}} = \mathrm{Spa}(A'_{Ig(p^\infty)}, A'^+_{Ig(p^\infty)}) \to \mathcal{X}^*(\epsilon)^{\mathrm{perf}},$$

we have $A'^+_{Ig(p^\infty)}/\varpi = \varinjlim A'^+_{Ig(p^n)}/\varpi$. The natural morphism of perfectoid $\mathcal{O}^a_{K^\flat}$-algebras $\mathcal{O}_{K^\flat}[[q^{1/p^\infty}]] \otimes_{A'^+} A'^+_{Ig(p^n)} \to \mathrm{Map}((\mathbb{Z}/p^n\mathbb{Z})^\times, \mathcal{O}_{K^\flat}[[q^{1/p^\infty}]])$ is an almost isomorphism because it is an isomorphism generically. Therefore, by tensoring with the almost flat A'^+-algebra $A'^+_{Ig(p^n)}$ and taking direct limits, we conclude that

$$A'^+_{Ig(p^\infty)}/\varpi \to \mathrm{Map}_{lc}(\mathbb{Z}_p^\times, \mathcal{O}_{K^\flat}/\varpi[[q^{1/p^\infty}]])$$

is almost injective. This implies that the map $S^\circ/p \to \mathrm{Map}_{lc}(\mathbb{Z}_p, \mathcal{O}_K/p[[q^{1/p^\infty}]])$ is almost injective, as we wanted to see. ∎

6.4 Tate parameter spaces on the good reduction locus

There is also a q-expansion principle for extending from the good reduction locus $\mathcal{X}^{gd}$ to $\mathcal{X}$.

Definition 6.13. Consider $\mathcal{O}_K((q))$ with the p-adic topology, that is we suppress the topology coming from q. Let $\mathcal{O}_K\langle\langle q\rangle\rangle := \mathcal{O}_K((q))^\wedge$ be the p-adic completion. These are the power series of the form

$$\mathcal{O}_K\langle\langle q\rangle\rangle = \{f = \sum_{n\in\mathbb{Z}} a_n q^n \mathcal{O}_K[[q^{\pm 1}]] \mid a_n \in \mathcal{O}_K \text{ such that } |a_n| \to 0 \text{ for } n \to -\infty\}.$$

Lemma 6.14. *The pair $(K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[q]], \mathcal{O}_K[[q]])$ considered with the p-adic topology is a sous-perfectoid Huber pair in the sense of [13]. In particular, the space $\overline{D} = \mathrm{Spa}(K \otimes_{\mathcal{O}_K} \mathcal{O}_K[[q]], \mathcal{O}_K[[q]])$ is a sous-perfectoid adic space. We have $\overline{D}(|q| < 1) = D$ and $\overline{D}(|q| \geq 1) = \mathrm{Spa}(K \otimes_{\mathcal{O}_K} \mathcal{O}_K\langle\langle q\rangle\rangle, \mathcal{O}_K\langle\langle q\rangle\rangle)$. These open subspaces cover $\overline{D}$ up to a rank 2-point.*

Proposition 6.15. *For every cusp there is a morphism $\overline{D} \to \mathcal{X}^*$.*

1. *On $D \subseteq \overline{D}$, this is an isomorphism onto $D \subseteq \mathcal{X}^*$.*
2. *The fibre over $\mathcal{X}^{gd} \subseteq \mathcal{X}^*$ is $\overline{D}(|q| \geq 1) = \mathrm{Spa}(K \otimes_{\mathcal{O}_K} \mathcal{O}_K\langle\langle q\rangle\rangle, \mathcal{O}_K\langle\langle q\rangle\rangle)$*
3. *The morphisms $\cup\overline{D} \to \mathcal{X}^*$ and $\mathcal{X}^{gd} \hookrightarrow \mathcal{X}^*$ cover $\mathcal{X}^*$.*

Theorem 6.16 (q-expansion principle IV)**.** *Let $c_0, \ldots, c_m$ be a collection of cusps of $\mathcal{X}^*$ such that each connected component contains at least one cusp. Then a function on $\mathcal{X}^{gd}$ extends to all of $\mathcal{X}^*$ if and only if its q-expansion at the Tate parameter space $\overline{D}(|q| \geq 1) \to \mathcal{X}^{gd}$ over each c_i is already in $\mathcal{O}_K[[q]] \otimes K \subseteq \mathcal{O}_K\langle\langle q\rangle\rangle$. In this case, the extension is unique.*

Proof. Restrict to $\mathcal{X}^*(0)$, then show that diagram is Cartesian by reducing mod p^n. □

References

[1] F. Andreatta, A. Iovita, and V. Pilloni. Le halo spectral. *Ann. Sci. Éc. Norm. Supér. (4)*, 51(3):603–655, 2018.

[2] C. Birkbeck, B. Heuer, and C. Williams. Overconvergent Hilbert modular forms via perfectoid modular varieties. *Preprint, available at https:arxiv.org/abs/1902.03985*, 2018.

[3] B. Conrad. Modular curves and rigid-analytic spaces. *Pure Appl. Math. Q.*, 2(1, Special Issue: In honor of John H. Coates. Part 1):29–110, 2006.

[4] B. Conrad. Arithmetic moduli of generalized elliptic curves. *J. Inst. Math. Jussieu*, 6(2):209–278, 2007.

[5] A. J. de Jong. Crystalline Dieudonné module theory via formal and rigid geometry. *Inst. Hautes Études Sci. Publ. Math.*, (82):5–96 (1996), 1995.

[6] B. Heuer. Tilting equivalences for modular forms. *In preparation, available upon request.*

[7] R. Huber. *Étale cohomology of rigid analytic varieties and adic spaces.* Aspects of Mathematics, E30. Friedr. Vieweg & Sohn, Braunschweig, 1996.

[8] N. M. Katz. p-adic properties of modular schemes and modular forms. In *Modular functions of one variable, III*, pages 69–190. Lecture Notes in Math., Vol. 350. Springer, Berlin, 1973.

[9] N. M. Katz and B. Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.

[10] P. Scholze. Perfectoid spaces. *Publ. Math. Inst. Hautes Études Sci.*, 116:245–313, 2012.

[11] P. Scholze. On torsion in the cohomology of locally symmetric varieties. *Ann. of Math. (2)*, 182(3):945–1066, 2015.

[12] P. Scholze and J. Weinstein. Moduli of p-divisible groups. *Camb. J. Math.*, 1(2):145–237, 2013.

[13] P. Scholze and J. Weinstein. Berkeley lectures on p-adic geometry. *Updated version, available at http://www.math.uni-bonn.de/people/scholze/Berkeley.pdf*, 2018.