HIGHER SEGAL STRUCTURES IN ALGEBRAIC K-THEORY

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Abstract. We introduce higher dimensional analogues of simplicial constructions due to Segal and Waldhausen, respectively producing the direct sum and algebraic K-theory spectra of an exact category. We interrelate them by totalizing, and investigate their fibrancy properties based on the formalism of higher Segal spaces of Dyckerhoff-Kapranov.

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Introduction

The notion of a Segal object was introduced by Rezk, motivated by the observation that a simplicial set satisfies the Segal conditions if and only if it is the nerve of a category. In this sense (and much more generally), they model the structure of an associative monoid.

The monoid structure comes from the correspondence

$$X_1 \times_{X_0} X_1 \xleftarrow{(\partial_2,\partial_0)} X_2 \xrightarrow{\partial_1} X_1, \tag{0.1}$$

which is a bona fide map if X is Segal. The definition of higher Segal objects is motivated by the question when (0.1) still satisfies the associativity constraints as a multi-valued operation. The answer is provided by the 2-Segal conditions, which were introduced in [11].

Namely, X is 2-Segal if X_n is recovered as a certain limit over its 2-skeleton in two different ways, described by the combinatorics of triangulations of the cyclic polygon on $n+1$ vertices.

As suggested in *op.cit.*, this allows for a natural extension to any dimension $d \geq 0$, based on the theory of d-dimensional cyclic polytopes and their triangulations. These higher Segal structures are introduced in this work. A categorical reason for cyclic polytopes to appear is that they provide a model for the free ω-categories generated by simplices.

Each cyclic polytope has two distinguished triangulations, obtained as projections of the lower and upper boundaries in the next higher dimension, which allow for a purely combinatorial description. The lower (resp. upper) d-Segal condition requires that X_n be recovered as the corresponding limit over its d-skeleton, for all $n \geq d$.

From a different perspective, unital 2-Segal spaces were defined and studied independently in [14] and its series of sequels under the name decomposition space. Further work in this area includes a precise description of unital 2-Segal sets in terms of double categories in [2], and the introduction of relative Segal conditions in [27] and [30], which model the structure

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of modules over multi-valued monoids. Moreover, in [28], unital 2-Segal spaces are shown to be equivalent to certain ∞-operads.

The first main objective of this work is to investigate the first examples of d-Segal objects, where $d > 2$, explicitly appearing in the literature. These simplicial categories naturally arise in the algebraic K-theory of exact categories $\mathcal{E}$. To wit, they are higher dimensional generalizations of Segal's construction $S_{\oplus}(\mathcal{E})$ in [25] and Waldhausen's construction $S(\mathcal{E})$ in [29], which provide deloopings of the direct sum and algebraic K-theory spectra of $\mathcal{E}$, respectively.

For $k = 2$, the k-dimensional Segal and Waldhausen constructions $S_{\oplus}^{\langle k\rangle}(\mathcal{E})$, resp. $S^{\langle k\rangle}(\mathcal{E})$, first appeared in the context of real algebraic K-theory as introduced by Hesselholt-Madsen [18], allowing the construction of genuine $\mathrm{Gal}(\mathbb{C}|\mathbb{R})$-equivariant deloopings in the presence of a duality structure on $\mathcal{E}$, which was further studied for example by Dotto [6].

The extension to arbitrary $k \geq 1$ is straight-forward; instead of a type of configuration space on the circle (in Segal's context) or the 2-sphere (Hesselholt-Madsen), we consider k-dimensional spheres. On the other hand, diagrams of short exact sequences (Waldhausen) and 2-extensions (Hesselholt-Madsen) are replaced by $(k + 2)$-term exact sequences.

Beyond being an evident generalization, $S^{\langle k\rangle}(\mathcal{E})$ also appears naturally on the simplicial side of the secondary Dold-Kan correspondence [10] in the case where $\mathcal{E}$ is a stable ∞-category, as a categorified Eilenberg-MacLane space.

It was already shown by Segal that $S_{\oplus}^{\langle 1\rangle}(\mathcal{E})$ satisfies the (lower 1-)Segal conditions, which generalize as follows.

Theorem 0.1. *The higher Segal construction $S_{\oplus}^{\langle k\rangle}(\mathcal{E})$ is a lower $(2k-1)$-Segal category.*

Similarly, for the Waldhausen construction, the 2-Segal property of $S^{\langle 1\rangle}(\mathcal{E})$ is one of the main results of [11]. Our first generalization of it puts some homological algebraic constraints on $\mathcal{E}$, as illustrated by the following statement.

Theorem 0.2. *The simplicial category $S^{\langle 2\rangle}(\mathcal{E})$ is upper 3-Segal if and only if $\mathcal{E}$ is an abelian category.*

Interestingly, this case is an outlier – in general, we only have the following slightly weaker result (any d-Segal property implies all higher dimensional ones).

Theorem 0.3. *The higher Waldhausen construction $S^{\langle k\rangle}(\mathcal{E})$ of the exact category $\mathcal{E}$ is $2k$-Segal.*

The main step in the proof for abelian $\mathcal{E}$ is the calculation of the path spaces of $S^{\langle k\rangle}(\mathcal{E})$ in order to make use of another inductive property of the Segal conditions; this also works for weakly idempotent complete $\mathcal{E}$. On the lowest simplicial level, this amounts to a version of the Noether isomorphism theorem for longer left exact sequences (or a higher octahedral axiom if $\mathcal{E}$ is derived; interestingly, this also applies to the theory of [19]).

From an algebraic perspective, our results lay the groundwork for further investigation of the algebraic structures induced by the Segal and Waldhausen constructions (cf. below). From a topological perspective, they are interesting since they provide new deloopings of the direct sum and algebraic K-theory of $\mathcal{E}$. Historically seen, much progress has been made due to the existence of new models which exhibit specific features more clearly than others, e.g. power operations in algebraic K-theory due to Grayson [15] and Nenashev [21].

Theorem 0.4. *There is a natural equivalence of simplicial categories from the k-dimensional Waldhausen construction of $\mathcal{E}$ to the total simplicial object of its k-fold S-construction,*

$$S^{\langle k\rangle}(\mathcal{E}) \xrightarrow{\simeq} TS^{(k)}(\mathcal{E}).$$

Similarly, there is an equivalence of simplicial categories

$$S_{\oplus}^{\langle k\rangle}(\mathcal{E}) \xrightarrow{\simeq} TS_{\oplus}^{(k)}(\mathcal{E}).$$

In particular, this induces equivalences on geometric realizations, from which we deduce the appropriate delooping and additivity theorems.

Corollary 0.5. (Delooping). *The algebraic K-theory spectrum of $\mathcal{E}$ is equivalent to*

$$|S^{\langle 0\rangle}(\mathcal{E})^{\simeq}| \longrightarrow \Omega|S^{\langle 1\rangle}(\mathcal{E})^{\simeq}| \longrightarrow \Omega^2|S^{\langle 2\rangle}(\mathcal{E})^{\simeq}| \longrightarrow \dots$$

where $(-)^{\simeq}$ denotes the maximal subgroupoid of a category.

(Additivity). *The functor (∂_2, ∂_0) on $S_2(\mathcal{E})$ induces a homotopy equivalence*

$$|S^{\langle k\rangle}(S_2(\mathcal{E}))^{\simeq}| \xrightarrow{\simeq} |S^{\langle k\rangle}(\mathcal{E})^{\simeq}| \times |S^{\langle k\rangle}(\mathcal{E})^{\simeq}|,$$

More precisely, the simplicial space $|S^{\langle k\rangle}(S_\bullet(\mathcal{E}))^{\simeq}|$ is a Segal space.

Let us briefly outline the structure of the paper. In §1, we summarize some basic theory of cyclic polytopes and their triangulations from [23] and [31], which is used in §2 to define and study relations between the higher Segal conditions from [12]. Novel results concern the interaction with total simplicial objects of multisimplicial objects.

In §3, we define the simplicial category $S_{\oplus}^{\langle k\rangle}(\mathcal{E})$, realize it as the total simplicial object of the iterated 1-dimensional construction, and prove Theorem 0.1. Section 4 provides definitions of the higher Waldhausen construction and first examples; we show that $S^{\langle k\rangle}(\mathcal{E})$ is the total simplicial object of Waldhausen's original construction, from which we deduce delooping and additivity theorems, as well as Theorem 0.2.

Finally, §5 introduces the requisite homological context enabling us to establish further Segal properties for the higher S-constructions; it also includes some further examples.

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1. Cyclic polytopes

In this section, we recall results on polytopes relevant to the study of higher Segal objects.

Definition 1.1. Let $d \geq 0$, and consider the moment curve

$$\gamma_d \colon \mathbb{R} \longrightarrow \mathbb{R}^d, \ t \longmapsto (t, t^2, t^3, \dots, t^d).$$

For a finite subset $N \subseteq \mathbb{R}$, the d-dimensional cyclic polytope on the vertices N is defined to be the convex hull of the set $\gamma_d(N) \subseteq \mathbb{R}^d$, and denoted by

$$C(N, d) = \operatorname{conv}(\gamma_d(N)).$$

The combinatorial type of the polytope $C(N, d)$ only depends on the cardinality of N. We will usually consider N to be the set $[n] = \{0, 1, \dots, n\}$, where $n \geq 0$.

Cyclic polytopes are simplicial polytopes, i.e., their boundary forms a simplicial complex, which organizes into two components (with non-empty intersection), as follows.

Definition 1.2. A point x in the boundary of $C([n], d+1)$ is called a lower point, if

$$(x - \mathbb{R}_{>0}) \cap C([n], d+1) = \emptyset,$$

where the half-line of positive real numbers $\mathbb{R}_{>0} \subseteq \mathbb{R}^{d+1}$ is embedded into the last coordinate. Similarly, x is said to be an upper point, if

$$(x + \mathbb{R}_{>0}) \cap C([n], d+1) = \emptyset.$$

The lower and upper points of the boundary form simplicial subcomplexes of $C([n], d+1)$, which admit the following purely combinatorial description.

Definition 1.3. Let $I \subseteq [n]$. A gap of I is a vertex $j \in [n] \setminus I$ in the complement of I. A gap is said to be even, resp. odd, if the cardinality $\#\{i \in I \mid i > j\}$ is even, resp. odd. The subset I is called even, resp. odd, if all gaps of I are even, resp. odd.

Proposition 1.4 (Gale's evenness criterion; [31], Theorem 0.7)**.** *Let $n \geq 0$, and let $I \subseteq [n]$ with $\#I = d{+}1$. Then I defines a d-simplex in the lower, resp. upper, boundary of $C([n], d{+}1)$ if and only if I is even, resp. odd.*

Forgetting the last coordinate of $\mathbb{R}^{d+1}$ defines a projection map

$$p\colon C([n], d+1) \longrightarrow C([n], d).$$

For any $I \subseteq [n]$ with $\#I - 1 = r \leq d$, the projection p maps the geometric r-simplex

$$|\Delta^I| \subseteq C([n], d+1)$$

homeomorphically to an r-simplex in $C([n], d)$.

Definition 1.5. The lower triangulation $\mathcal{T}_\ell$ of the polytope $C([n], d)$ is the triangulation given by the projections under p of the simplices contained in the lower boundary of $C([n], d+1)$. Similarly, the upper triangulation $\mathcal{T}_u$ of $C([n], d)$ is defined by the projections of the simplices contained in the upper boundary of the polytope $C([n], d+1)$.

Vice versa, any triangulation of $C([n], d)$ induces a piecewise linear section of p, whose image defines a simplicial subcomplex of $C([n], d+1)$. This interplay between the cyclic polytopes in different dimensions is what makes their combinatorics comparatively tractible.

Definition 1.6. Given a set $\mathcal{I} \subseteq 2^{[n]}$ of subsets of $[n]$, we denote by

$$\Delta^{\mathcal{I}} \subseteq \Delta^n \tag{1.1}$$

the simplicial subset of Δ^n whose m-simplices are given by those maps $[m] \to [n]$ which factor over some $I \in \mathcal{I}$.

From the above discussion, it follows that we have canonical homeomorphisms

$$|\Delta^{\mathcal{T}_\ell}| \xrightarrow{\ \sim\ } C([n], d), \text{ and } |\Delta^{\mathcal{T}_u}| \xrightarrow{\ \sim\ } C([n], d),$$

expressing the lower, resp. upper, triangulation of $C([n], d)$ geometrically.

Definition 1.7. Let $I, J \subseteq [n]$ be subsets of cardinality $d+1$, as well as $|\Delta^I|$ and $|\Delta^J|$ the geometric d-simplices in $C([n], d) \subseteq \mathbb{R}^d$ they define, respectively. Let $L(|\Delta^J|)$ denote the set of lower boundary points of $|\Delta^J| = C(J, d)$, and similarly, let $U(|\Delta^I|)$ be the set of upper boundary points of $|\Delta^I| = C(I, d)$. We write

$$|\Delta^I| \prec |\Delta^J| \iff |\Delta^I| \cap |\Delta^J| \subseteq U(|\Delta^I|) \cap L(|\Delta^J|).$$

If $|\Delta^I| \prec |\Delta^J|$, then we say that $|\Delta^I|$ lies below the simplex $|\Delta^J|$.

Proposition 1.8 ([23], Corollary 5.9)**.** *The transitive closure of $\prec$ defines a partial order on the set of nondegenerate d-simplices in Δ^n.*

Remark 1.9. Suppose that $\mathcal{T} \subseteq 2^{[n]}$ consists of subsets of $[n]$ of cardinality $d+1$ and defines a triangulation of the cyclic polytope $C([n], d)$. In particular,

$$|\Delta^{\mathcal{T}}| \cong C([n], d).$$

Let $I_0 \in \mathcal{T}$. Then either $L(|\Delta^{I_0}|)$ is contained in $L(|\Delta^{\mathcal{T}}|)$, or there is some $I_1 \in \mathcal{T}$ such that the simplex $|\Delta^{I_0}|$ lies below $|\Delta^{I_1}|$. Iterating this argument, we obtain a chain

$$|\Delta^{I_0}| \prec |\Delta^{I_1}| \prec |\Delta^{I_2}| \prec \dots$$

of subsimplices of $|\Delta^{\mathcal{T}}|$. The statement of Proposition 1.8 implies that this chain is acyclic and therefore has to terminate after finitely many steps. Thus, there exists $I_\infty \in \mathcal{T}$ with

$$L(|\Delta^{I_\infty}|) \subseteq L(|\Delta^{\mathcal{T}}|).$$

Similarly, there exists a set $I_{-\infty} \in \mathcal{T}$ such that $U(|\Delta^{I_{-\infty}}|) \subseteq U(|\Delta^{\mathcal{T}}|)$.

2. Higher Segal conditions

Let $\mathcal{C}$ be an ∞-category which admits finite limits. Following [12], we introduce a framework of fibrancy properties of simplicial objects in $\mathcal{C}$ governed by the combinatorics from §1 of cyclic polytopes and their triangulations.

Definition 2.1. For $n \geq d \geq 0$, we introduce the lower subposet of $2^{[n]}$ as follows;

$$\mathcal{L}([n], d) = \{J \mid J \subseteq I \text{ for some even } I \subseteq [n] \text{ with } \#I = d+1\}.$$

Analogously, we define

$$\mathcal{U}([n], d) \subseteq 2^{[n]}$$

as the poset of all subsets contained in an odd subset $I \subseteq [n]$ of cardinality $\#I = d+1$.

By Proposition 1.4, the sets of subsimplices of $C([n], d)$ described by $\mathcal{L}([n], d)$ and $\mathcal{U}([n], d)$ define the lower and upper triangulations of the cyclic polytope, respectively.

Definition 2.2. Let $d \geq 0$. A simplicial object $X \in \mathcal{C}_\Delta$ is called

- lower d-Segal if, for every $n \geq d$, the natural map
$$X_n \longrightarrow \varprojlim_{I \in \mathcal{L}([n],d)} X_I$$
is an equivalence;
- upper d-Segal if, for every $n \geq d$, the natural map
$$X_n \longrightarrow \varprojlim_{I \in \mathcal{U}([n],d)} X_I$$
is an equivalence;
- d-Segal if X is both lower and upper d-Segal.

Example 2.3. Let $X \in \mathcal{C}_\Delta$ be a simplicial object.

(1) Then X is lower (or upper) 0-Segal if and only if $X \xrightarrow{\simeq} X_0$ is equivalent to the constant simplicial object on its 0-cells.

(2) The simplicial object X is lower 1-Segal if, for every $n \geq 1$, the map
$$X_n \longrightarrow X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \times_{X_{\{2\}}} \cdots \times_{X_{\{n-1\}}} X_{\{n-1,n\}}$$
is an equivalence. That is to say, X is a Segal object in the sense of Rezk [24].

For $X \in \mathrm{Set}_\Delta$, this means that X is equivalent to the nerve of the category with objects X_0, morphisms X_1, and composition induced by the correspondence
$$X_{\{0,1\}} \times_{X_{\{1\}}} X_{\{1,2\}} \xleftarrow{\sim} X_2 \longrightarrow X_{\{0,2\}}.$$
Furthermore, $X \in \mathrm{Set}_\Delta$ is 1-Segal if and only if this defines a discrete groupoid. In fact, an upper 1-Segal set is a quiver; in general, an object $X \in \mathcal{C}_\Delta$ is upper 1-Segal if, for every $n \geq 1$, we have
$$X_n \xrightarrow{\simeq} X_{\{0,n\}}.$$

(3) The simplicial object X is lower 2-Segal if, for every $n \geq 2$, the map
$$X_n \longrightarrow X_{\{0,n-1,n\}} \times_{X_{\{0,n-1\}}} X_{\{0,n-2,n-1\}} \times_{X_{\{0,n-2\}}} \cdots \times_{X_{\{0,2\}}} X_{\{0,1,2\}}$$
is an equivalence. Similarly, X is upper 2-Segal if, for every $n \geq 2$, we have
$$X_n \xrightarrow{\simeq} X_{\{0,1,n\}} \times_{X_{\{1,n\}}} X_{\{1,2,n\}} \times_{X_{\{2,n\}}} \cdots \times_{X_{\{n-2,n\}}} X_{\{n-2,n-1,n\}}.$$
It follows that X is 2-Segal if and only if it is 2-Segal in the sense of [11]. This is most readily seen by reducing to Segal objects as in (2) by applying the respective path space criteria, Proposition 2.8 and [11], Theorem 6.3.2 – or by Proposition 2.6.

Lemma 2.4. *Let $d \geq 0$. A limit of lower, resp. upper, d-Segal objects is again lower, resp. upper, d-Segal.*

Proof. This is clear, because all the conditions are defined in terms of finite limits. □

Remark 2.5. Let X be a simplicial object in an ∞-category $\mathcal{C}$ which admits limits. Then we can form the right Kan extension of X along the (opposite of the) Yoneda embedding

$$\mathcal{Y}^{\mathrm{op}}\colon N(\Delta^{\mathrm{op}}) \longrightarrow \mathrm{Fun}(N(\Delta^{\mathrm{op}}), \mathrm{Top})^{\mathrm{op}},$$

where Top denotes the ∞-category of spaces. In particular, by means of this extension, we may evaluate X on any simplicial set. Then we can reformulate the higher Segal conditions as follows. Let $\Sigma_d = \Sigma_{\ell|d} \cup \Sigma_{u|d}$ denote the collection of d-Segal coverings, where

$$\Sigma_{\ell|d} = \{\Delta^{\mathcal{L}([n],d)} \hookrightarrow \Delta^n \mid n \geq d\}, \text{ and } \Sigma_{u|d} = \{\Delta^{\mathcal{U}([n],d)} \hookrightarrow \Delta^n \mid n \geq d\},$$

see [11], §5.2. Then X is d-Segal if and only if it is Σ_d-local (cf. *loc.cit.*, Proposition 5.1.4). In this case, we say that the elements of Σ_d are X-equivalences (meaning, X maps them to equivalences in $\mathcal{C}$).

Similarly, X is lower, resp. upper, d-Segal if and only if it is $\Sigma_{\ell|d}$-local, resp. $\Sigma_{u|d}$-local.

Proposition 2.6. *Let $d \geq 0$, and let X be a simplicial object in an ∞-category $\mathcal{C}$ with limits. Then X is d-Segal if and only if, for every $n \geq d$, and every triangulation of the cyclic polytope $C([n], d)$ defined by the poset of simplices $\mathcal{T} \subseteq 2^{[n]}$, the natural map*

$$X_n \longrightarrow \varprojlim_{I \in \mathcal{T}} X_I$$

is an equivalence.

Proof. By [23], Corollary 5.12, any triangulation $\mathcal{T}$ of $C([n], d)$ can be connected to the lower and upper triangulations $\mathcal{T}_\ell$ and $\mathcal{T}_u$ via sequences of elementary flips of the form

$$L(|\Delta^I|) \subseteq |\Delta^I| \supseteq U(|\Delta^I|).$$

This implies that we have a zig-zag of X-equivalences of the form

$$\Delta^{\mathcal{L}([n],d)} \longrightarrow \ldots \longleftarrow \Delta^{\mathcal{T}} \longrightarrow \ldots \longleftarrow \Delta^{\mathcal{U}([n],d)}$$

in the category of simplicial sets over Δ^n. This implies that the inclusion $\Delta^{\mathcal{T}} \subseteq \Delta^n$ is again an X-equivalence by 2-out-of-3, which was to be shown.

For the converse, there is nothing to prove. □

Definition 2.7. Let X be a simplicial object in an ∞-category $\mathcal{C}$. The left path space $P^{\triangleleft}X$ is the simplicial object in $\mathcal{C}$ defined as the pullback of X along the endofunctor

$$c^{\triangleleft}\colon \Delta \to \Delta,\ [n] \mapsto [0] \oplus [n].$$

Here, for two linearly ordered sets I and J, the ordinal sum $I \oplus J$ is the disjoint union $I \amalg J$ of sets, endowed with the linear order where $i \leq j$ for every pair of $i \in I$ and $j \in J$.

Similarly, the right path space $P^{\triangleright}X$ is given by the pullback of X along

$$c^{\triangleright}\colon \Delta \to \Delta,\ [n] \mapsto [n] \oplus [0].$$

Proposition 2.8 (Path space criterion)**.** *Let $d \geq 0$, and let $\mathcal{C}$ be an ∞-category with finite limits. Let $X \in \mathcal{C}_\Delta$ be a simplicial object.*

- *Suppose d is even. Then*
  1. *X is lower d-Segal if and only if $P^{\triangleleft}X$ is lower $(d-1)$-Segal,*
  2. *X is upper d-Segal if and only if $P^{\triangleright}X$ is lower $(d-1)$-Segal.*
- *Suppose d is odd. Then the following conditions are equivalent.*
  - (*i*) *X is upper d-Segal.*
  - (*ii*) *$P^{\triangleleft}X$ is upper $(d-1)$-Segal.*
  - (*iii*) *$P^{\triangleright}X$ is lower $(d-1)$-Segal.*

Proof. We show that if d is even, then X is an upper d-Segal object if and only if $P^{\triangleright}X$ is lower $(d-1)$-Segal. All other assertions follow by analogous arguments.

Let $\ell([n], d)$ denote the set of maximal elements of the poset $\mathcal{L}([n], d)$, so that

$$\Delta^{\mathcal{L}([n],d)} = \bigcup_{I \in \ell([n],d)} \Delta^I$$

is the minimal presentation, and similarly for $u([n],d) \subseteq \mathcal{U}([n],d)$.

Then the claim is an immediate consequence of the following observation. A subset $I \subseteq [n]$ of cardinality $d+1$ is odd if and only if $n \in I$ and $I \setminus \{n\}$ is an even subset of $[n-1]$. Thus the map $c^{\rhd} \colon \ell([n-1],d-1) \to u([n],d)$, $I \mapsto I \oplus [0]$, is a bijection. $\square$

Remark 2.9. There is no path space criterion for lower d-Segal objects if d is odd. While

$$\ell([5],3) = \{\{0,1,2,3\},\{0,1,3,4\},\{0,1,4,5\},\{1,2,4,5\},\{2,3,4,5\},\{1,2,3,4\}\},$$

the maps $c^{\lhd} \colon \ell([4],2) \to \ell([5],3)$, $I \mapsto [0] \oplus I$, and $c^{\rhd} \colon u([4],2) \to \ell([5],3)$, $I \mapsto I \oplus [0]$, are not even jointly surjective, and their images

$$\mathrm{im}(c^{\lhd}) = \{\{0,1,2,3\},\{0,1,3,4\},\{0,1,4,5\}\},\ \mathrm{im}(c^{\rhd}) = \{\{0,1,4,5\},\{1,2,4,5\},\{2,3,4,5\}\}$$

intersect. Rather, the complement of $\mathrm{im}(c^{\rhd})$ in $\ell([n],d)$ is given by $\ell([n-1],d)$, for all $d \in \mathbb{N}$ and $n \geq d$. By induction, we obtain a disjoint decomposition of sets

$$\ell([n],d) = \coprod_{j=d}^{n} u([j-1],d-1) \oplus \{j\}. \tag{2.1}$$

In fact, this statement, resp. its dual, also follow by grouping the elements of $\ell([n],d)$ by their maximal, resp. minimal, vertex, and the following explicit description. If d is odd, then

$$\ell([n],d) = \Big\{ \coprod_{i \in J} \{i, i+1\} \;\Big|\; J \subseteq [n] \text{ with } \#J = \frac{d+1}{2} \text{ and } |i-j| > 1 \text{ for all } i,j \in I \Big\}, \tag{2.2}$$

for all $n \geq d$. All other $\ell([n],d)$ and $u([n],d)$ are described from this by applying the proof of Proposition 2.8.

Proposition 2.10. *Let X be a simplicial object in an ∞-category $\mathcal{C}$ which admits limits. Assume that X is lower or upper d-Segal. Then X is $(d+1)$-Segal.*

Proof. We show the statement assuming that X is lower d-Segal. The proof for upper d-Segal objects is similar. Let $n \geq d+1$ and consider a collection $\mathcal{T}$ defining a triangulation $|\Delta^{\mathcal{T}}|$ of the cyclic polytope $C([n],d+1)$. Recall that $\mathcal{T}_\ell$ defines the simplicial complex $L(|\Delta^{\mathcal{T}}|)$ of lower facets, and the projection $p \colon C([n],d+1) \to C([n],d)$ identifies $|\Delta^{\mathcal{T}_\ell}| \subseteq C([n],d+1)$ with the simplicial subcomplex defining the lower triangulation

$$p(|\Delta^{\mathcal{T}_\ell}|) \subseteq C([n],d).$$

Thus, we obtain a commutative diagram

$$\begin{tikzcd} \Delta^{\mathcal{T}} \arrow[r, "\iota"] & \Delta^n \\ \Delta^{\mathcal{T}_\ell} \arrow[u, "\kappa"] \arrow[ur, "\iota_\ell"'] & \end{tikzcd} \tag{2.3}$$

of simplicial sets, in which by definition, $\iota \in \Sigma_{d+1}$ and $\iota_\ell \in \Sigma_{\ell|d}$. In order to deduce $(d+1)$-Segal descent for X, we have to show that $\iota \in \overline{\Sigma}_{\ell|d}$, the collection of $\Sigma_{\ell|d}$-equivalences, by the tautological part of Proposition 2.6.

By the 2-out-of-3 property of $\overline{\Sigma}_{\ell|d}$, it suffices to show that $\kappa \in \overline{\Sigma}_{\ell|d}$. We will do so by showing that κ can be obtained as an iterated pushout along morphisms in $\Sigma_{\ell|d}$.

By Remark 1.9, the triangulation $|\Delta^{\mathcal{T}}|$ of $C([n],d+1)$ contains a maximal $(d+1)$-simplex of the form $|\Delta^I|$, defined by a singleton collection $\{I\} \subseteq \binom{[n]}{d+2}$. Let $\mathcal{I}_\ell$ be the set which defines the lower facets of $|\Delta^I|$, defining a triangulation $|\Delta^{\mathcal{I}_\ell}|$ of $C(I,d)$. Then the inclusion of simplicial sets

$$\kappa_\ell \colon \Delta^{\mathcal{I}_\ell} \longrightarrow \Delta^I$$

is contained in $\Sigma_{\ell|d}$. Further, since $|\Delta^I|$ is a maximal simplex, we have a pushout diagram

$$\begin{tikzcd} \Delta^{\mathcal{T}} & \Delta^I \arrow[l, hook'] \\ \Delta^{\mathcal{T}^{(1)}} \arrow[u, hook, "\kappa^{(1)}"] \arrow[ur, phantom, "\lrcorner"] & \Delta^{\mathcal{I}_\ell} \arrow[l, hook'] \arrow[u, hook, "\kappa_\ell"'] \end{tikzcd}$$

of simplicial sets, where $\mathcal{T}^{(1)} = \mathcal{T} \smallsetminus \{I\}$. Thus, the map $\kappa^{(1)}$ lies in $\overline{\Sigma}_{\ell|d}$, and $|\Delta^{\mathcal{T}^{(1)}}|$ is an admissible simplicial subcomplex of $C([n], d+1)$ with one $(d+1)$-simplex less than $|\Delta^{\mathcal{T}}|$.

Assume that the triangulation $|\Delta^{\mathcal{T}}|$ consists of exactly r simplices of dimension $(d+1)$. By iterating the argument just given, we obtain a chain of morphisms

$$\Delta^{\mathcal{T}_\ell} \hookrightarrow \Delta^{\mathcal{T}^{(r-1)}} \hookrightarrow \ldots \hookrightarrow \Delta^{\mathcal{T}^{(1)}} \hookrightarrow \Delta^{\mathcal{T}}$$

in $\overline{\Sigma}_{\ell|d}$ whose composite is the morphism κ from (2.3). Thus, it is also contained in $\overline{\Sigma}_{\ell|d}$. □

Definition 2.11. Let $\mathcal{C}$ be an ∞-category with finite limits, and $Y \in \mathcal{C}_{\Delta^{\times k}}$ a k-fold simplicial object, for $k \geq 0$. We will denote by $DY \in \mathcal{C}_\Delta$ the diagonal of Y, which is defined to be the pullback of Y under the diagonal embedding $\mathrm{diag}\colon \Delta \hookrightarrow \Delta^{\times k}$.

Remark 2.12. Let $\mathcal{C}$ be an ∞-category with finite limits, and let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a k-fold simplicial object, and let $r \geq 0$. We conjecture the following compatibilities.

(1) If Y is k-fold lower $(2r-1)$-Segal, $r \geq 1$, then DY is lower $(2kr-1)$-Segal.
(2) If Y is k-fold lower, resp. upper, $2r$-Segal, then DY is upper, resp. lower, $2kr$-Segal.
(3) If Y is k-fold upper $(2r+1)$-Segal, then DY is upper $(2kr+1)$-Segal.

Note that (1) implies the other two statements, by repeated applications of the path space criterion; indeed, $P^{\triangleleft}DY \simeq DP^{\triangleleft}_{(k)}Y$ and $P^{\triangleright}DY \simeq DP^{\triangleright}_{(k)}Y$, where the notation $P^{\triangleleft}_{(k)}$ and $P^{\triangleright}_{(k)}$ means application of the corresponding path space functor in each variable.

Definition 2.13. Let $V\colon \mathcal{C}_\Delta \to \mathcal{C}_{\Delta^{\times k}}$ be the total décalage, which is the pullback along the (k-fold) ordinal sum $\oplus\colon \Delta^{\times k} \to \Delta$. The total simplicial object of $Y \in \mathcal{C}_{\Delta^{\times k}}$ is denoted by

$$TY \in \mathcal{C}_\Delta,$$

where T is defined as the right adjoint of V.

Lemma 2.14. *Let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a k-fold simplicial object. Then for all $n \geq 0$, the n-cells of the total simplicial object of Y are computed by*

$$(TY)_n \xrightarrow{\simeq} \varprojlim_{\substack{I_1 \cup \ldots \cup I_k = [n] \\ I_1 \leq \ldots \leq I_k}} Y_{I_1, \ldots, I_k}. \tag{2.4}$$

Proof. By induction, we can reduce to $k = 2$. On [26], p.10, we find the equivalent formula

$$(TY)_n \xrightarrow{\simeq} \mathrm{eq}\Big(\prod_{i=0}^{n} Y_{\{0,\ldots,i\},\{i,\ldots,n\}} \overset{\psi^{\triangleleft}}{\underset{\chi^{\triangleright}}{\rightrightarrows}} \prod_{I \oplus J = [n]} Y_{I,J} \Big),$$

where the components of $\psi^{\triangleleft}$ are given by $\partial_{\bullet,0} \circ \mathrm{pr}_i$ for $I = \{0, \ldots, i\}$, and $\chi^{\triangleright}$ consists of the functors $\partial_{i,\bullet} \circ \mathrm{pr}_i$ for $J = \{i, \ldots, n\}$ (also cf. [1], §III), but without proof.

Thus we spell out the relevant cofinality argument here. Namely, the full pointwise Kan extension is indexed by the category of pairs (I, J) with a map $I \oplus J \to [n]$; we claim that the (in fact, full) subcategory on all (I, J) with either $I \oplus J \xrightarrow{=} [n]$ or $I \oplus J \to I \vee J = [n]$, with the only morphisms given by $(I \smallsetminus (I \cap J), J) \to (I, J)$ and $(I, J \smallsetminus (I \cap J)) \to (I, J)$, is cofinal.

Indeed, given an arbitrary $(I, J) \in \Delta \times \Delta$ with $f : I \oplus J \to [n]$, in case $I \cap J \neq \emptyset$, we necessarily have $f(I \cap J) = \{i\}$ for some $0 \leq i \leq n$. Then there exists a unique morphism

$$(I, J) \to (\{0, \ldots, i\}, \{i, \ldots, n\})$$

with target in our subcategory. On the other hand, if $I \cap J = \emptyset$, let $i < j$ be the maximal element of I and minimal element of J, respectively. Then we have the following contractible choice of targets in our subcategory,

$$\begin{tikzcd}
(\{0,\dots,i\},\{i+1,\dots,n\}) \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & (\{0,\dots,j-1\},\{j,\dots,n\}). \arrow[dl] \\
& (\{0,\dots,i\},\{i,\dots,n\}) & \dots & (\{0,\dots,j-1\},\{j-1,\dots,n\}) &
\end{tikzcd}$$

□

Definition 2.15. Let $\underline{d} = (d_i) \in \mathbb{N}^k$, and let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a k-fold simplicial object. We say that Y is (lower, resp. upper) $\underline{d}$-Segal if it is (lower, resp. upper) d_i-Segal in the ith simplicial direction, for all $1 \leq i \leq k$.

Proposition 2.16. *Let $\mathcal{C}$ be an ∞-category with finite limits, $\underline{r} \in \mathbb{N}^k$, and let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a lower $(2\underline{r}-1)$-Segal object. Then $TY \in \mathcal{C}_\Delta$ is lower $(2r-1)$-Segal, where $r = \sum r_i$.*

Proof. For $k = 1$, this is a tautology. Now assume the statement for some $k \geq 1$, and consider a lower $(2\underline{r}-1)$-Segal object $Y = Y_{\bullet,\bullet} \in (\mathcal{C}_\Delta)_{\Delta^{\times k}}$ in $\mathcal{C}_\Delta$ which is lower $(2s-1)$-Segal in the remaining coordinate. By induction, $TY_{m,\bullet}$ is lower $(2r-1)$-Segal, for all $m \geq 0$, and we can reduce to the case $k = 2$. Thus, we want to prove that, for all $n \geq 2(r+s)-1$, the inclusion

$$\bigcup_{H \in \ell} \bigcup_{\substack{H = I \vee J \\ I \leq J}} \Delta^I \times \Delta^J \hookrightarrow \bigcup_{i=0}^{n} \Delta^{\{0,\dots,i\}} \times \Delta^{\{i,\dots,n\}} \tag{2.5}$$

is a Y-equivalence, where $\ell = \ell([n], 2(r+s)-1)$, and $I \vee J$ signifies the non-disjoint union. By eliminating redundant summands, we can write the left-hand side as

$$\bigcup_{H \in \ell} \Bigg(\bigcup_{\substack{H = I \vee J,\ I \leq J \\ 2r \leq \# I \leq 2r+1}} \Delta^I \times \Delta^J \cup \bigcup_{i=0}^{2r-1} \Delta^{[i]} \times \Delta^{H \smallsetminus [i-1]} \cup \bigcup_{j=n-2s}^{n-1} \Delta^{H \cap [j+1]} \times \Delta^{[n] \smallsetminus [j]} \Bigg).$$

Combining this, as well as the lower $(2s-1)$-Segal property in the other variable, with Proposition 2.10, and using (2.2), the statement follows. □

Remark 2.17. Let $\mathcal{C} = \mathrm{Set}$. Then the functor defined by Proposition 2.16,

$$T\colon \{k\text{-fold categories}\} \longrightarrow \{\text{lower } (2k-1)\text{-Segal sets}\} \tag{2.6}$$

is not an equivalence. In fact, for $k = 2$, the main result of [2] shows that the functor

$$V\colon \{2\text{-Segal sets}\} \longrightarrow \{\text{double categories}\}$$

induces an equivalence between the subcategory of unital 2-Segal sets on the left- and stable double categories together with the extra datum of an augmentation on the right-hand side. In particular, this implies that (2.6) cannot be fully faithful.

It is an interesting problem to generalize the above equivalence to arbitrary k. The definition of stability directly extends to k-fold categories (in terms of $(k+1)$-cubes), and so does the notion of augmentation; the analogue of unitality for higher Segal objects should involve degenerate triangulations of the appropriate cyclic polytopes.

Our considerations in §4 suggest a candidate for the inverse functor; on the other hand, it appears that for V to produce k-fold Segal objects, it still needs to be restricted to 2-Segal objects.

Our next result is the analogue of Remark 2.12 (2) for the total simplicial object; however, the proof is not as straight-forward, for the following reason.

Lemma 2.18. *Let $\mathcal{C}$ be an ∞-category with finite limits, and let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a k-fold simplicial object, $k \geq 1$. Then $P^{\triangleleft} TY \simeq TP^{\triangleleft} Y$ and $P^{\triangleright} TY \simeq TP^{\triangleright} Y$, where $P^{\triangleleft}$, resp. $P^{\triangleright}$, is applied in the first, resp. last, simplicial direction.*

Proof. This is an immediate consequence of the following base change squares,

$$
\begin{array}{ccc}
\Delta^{\times k} & \xrightarrow{c^{\triangleleft}\times \mathrm{id}} & \Delta^{\times k} \\
{\scriptstyle\oplus}\big\downarrow & & \big\downarrow{\scriptstyle\oplus} \\
\Delta & \xrightarrow{c^{\triangleleft}} & \Delta
\end{array}
\qquad\qquad
\begin{array}{ccc}
\Delta^{\times k} & \xrightarrow{\mathrm{id}\times c^{\triangleright}} & \Delta^{\times k} \\
{\scriptstyle\oplus}\big\downarrow & & \big\downarrow{\scriptstyle\oplus} \\
\Delta & \xrightarrow{c^{\triangleright}} & \Delta
\end{array}
$$

where $c^{\triangleleft}$ and $c^{\triangleright}$ are the maps from Definition 2.7. □

Proposition 2.19. *Let $\mathcal{C}$ be an ∞-category with finite limits. Let $k \geq 1$, and let $Y \in \mathcal{C}_{\Delta^{\times k}}$ be a $2\underline{r}$-Segal object, $\underline{r} \in \mathbb{N}^k$. Then $TY \in \mathcal{C}_\Delta$ is a $2r$-Segal object, where $r = \sum r_i$.*

Proof. For $k = 1$, there is nothing to show. In order to verify the upper $2k$-Segal condition by induction, we use the path space criterion as well as (2.1) to obtain

$$u([n], 2k) = \ell([n-1], 2k-1) \oplus \{n\} = \coprod_{j=2k-1}^{n-1} u([j-1], 2k-2) \oplus \{j, n\}. \tag{2.7}$$

This suffices, since for the lower $2k$-Segal condition, we can use the dual decomposition,

$$\ell([n], 2k) = \coprod_{i=1}^{n-2k+1} \{0, i\} \oplus \ell([n] \smallsetminus [i], 2k-2).$$

Now let $k = 2$. We aim to exhibit the n-cells of the total simplicial object as the limit

$$(TY)_n \xrightarrow{\simeq} \varprojlim_{H \in \mathcal{U}([n],4)} \varprojlim_{\substack{I \cup J = H \\ I \leq J}} Y_{I,J}.$$

By (2.7), the $H \in u([n], 4)$ are precisely $H = \{i-1, i, j-1, j, n\}$ with $0 < i < j-1 < n-1$. The corresponding factor in the limit is of the following form,

$$\begin{aligned} &Y_{(i-1)i,i(j-1)jn} \times_{Y_{(i-1)i,(j-1)jn}} Y_{(i-1)i(j-1),(j-1)jn} \times_{Y_{(i-1)i(j-1),jn}} Y_{(i-1)i(j-1)j,jn} \\ \simeq\ &Y_{(i-1)i,i(j-1)n} \times_{Y_{(i-1)i,(j-1)n}} Y_{(i-1)i(j-1),(j-1)jn} \times_{Y_{(i-1)(j-1),jn}} Y_{(i-1)(j-1)j,jn} \end{aligned} \tag{2.8}$$

by the upper 2-Segal property in the last and the lower 2-Segal property in the first coordinate, respectively. But the factors of the form $Y_{(i-1)i,i(j-1)n}$ and $Y_{(i-1)(j-1)j,jn}$ cancel precisely with the non-maximal elements of $\mathcal{U}([n], 4)$, with the exception of the extremal cases $Y_{01,12n}$ and $Y_{0(n-2)(n-1),(n-1)n}$, respectively.

On the other hand, we can describe the n-cells of the total simplicial object as follows,

$$\begin{aligned} (TY)_n \simeq\ &Y_{01,1\ldots n} \times_{Y_{01,2\ldots n}} Y_{012,2\ldots n} \times_{Y_{012,3\ldots n}} \cdots \times_{Y_{0\ldots(n-3),(n-2)(n-1)n}} \\ &Y_{0\ldots(n-2),(n-2)(n-1)n} \times_{Y_{0\ldots(n-2),(n-1)n}} Y_{0\ldots(n-1),(n-1)n} \\ \simeq\ &Y_{01,12n} \times_{Y_{01,2n}} Y_{012,23n} \times_{Y_{012,3n}} \cdots \times_{Y_{0\ldots(n-3),(n-2)(n-1)n}} \\ &Y_{0\ldots(n-2),(n-2)(n-1)n} \times_{Y_{0(n-2),(n-1)n}} Y_{0(n-2)(n-1),(n-1)n} \end{aligned} \tag{2.9}$$

by applying the upper 2-Segal property in the first variable wherever possible, as well as the lower 2-Segal property in the last variable to the final factor. But the factor $Y_{0\ldots(j-1),(j-1)jn}$ is precisely the limit of the (2.8) after cancellation for all $0 < i < j-1$. □

Definition 2.20. The edgewise subdivision functor $E\colon \mathcal{C}_\Delta \to \mathcal{C}_\Delta$ is the pullback along

$$e\colon \Delta \longrightarrow \Delta,\ [n] \longmapsto [n]^{\mathrm{op}} \oplus [n].$$

Remark 2.21. By the work [3] of Bergner, Osorno, Ozornova, Rovelli, and Scheimbauer, a simplicial object $X \in \mathcal{C}_\Delta$ is 2-Segal if and only if $EX \in \mathcal{C}_\Delta$ is Segal. Again, it would be very interesting to find an analogous result for the higher Segal conditions.

3. The higher Segal construction

Let $\mathcal{D}$ be a pointed category with finite products. In this section, we study a generalization of a construction due to Segal [25] which is similar to a construction proposed (for $k = 2$) by Hesselholt and Madsen [18]. The higher dimensional variants (for $k \geq 3$) are straightforward to define, but do not seem to have appeared in the literature as of yet.

Let $\mathrm{Fin}_\times$ denote the category of finite pointed sets. For $T \in \mathrm{Fin}_\times$, we denote by $\mathcal{P}(T)$ its poset of pointed subsets, considered as a small category.

Definition 3.1. Let $T \in \mathrm{Fin}_\times$ be a finite pointed set. A $\mathcal{D}$-valued presheaf

$$\mathcal{F}\colon \mathcal{P}(T)^{\mathrm{op}} \longrightarrow \mathcal{D}$$

on $\mathcal{P}(T)$ is called a sheaf if, for every pointed subset $U \subseteq T$, the canonical map

$$\mathcal{F}(U) \longrightarrow \prod_{u \in U \smallsetminus \{*\}} \mathcal{F}(\{*, u\})$$

is an isomorphism. We denote by $\mathrm{Sh}(T, \mathcal{D})$ the category of $\mathcal{D}$-valued sheaves on $\mathcal{P}(T)$.

Given a map $\rho\colon T \to T'$ in $\mathrm{Fin}_\times$, we define the pointed preimage functor

$$\rho^\times\colon \mathcal{P}(T') \longrightarrow \mathcal{P}(T),\ U \longmapsto \rho^{-1}(U \smallsetminus \{*\}) \amalg \{*\}.$$

Then the direct image functor $\mathcal{F} \mapsto \rho_*\mathcal{F} = \mathcal{F} \circ \rho^\times$ makes the assignment

$$\mathrm{Sh}(-, \mathcal{D})\colon \mathrm{Fin}_\times \longrightarrow \mathrm{Cat}$$

into a functor with values in the category of small categories.

Definition 3.2. Let $k \geq 1$. The k-dimensional Segal construction of $\mathcal{D}$ is defined to be the simplicial category

$$S_\oplus^{\langle k \rangle}(\mathcal{D}) = \mathrm{Sh}(S^k, \mathcal{D}) \in \mathrm{Cat}_\Delta,$$

where $S^k = \Delta^k/\partial\Delta^k$ is considered as a simplicial object in $\mathrm{Fin}_\times$.

Lemma 3.3. *Let $0 \leq i \leq n$. The face map $\partial_i\colon \mathrm{Sh}(S^k_n, \mathcal{D}) \to \mathrm{Sh}(S^k_{n-1}, \mathcal{D})$ is an isofibration.*

Proof. Let $\rho_i\colon S^k_n \to S^k_I$ be the corresponding map in $\mathrm{Fin}_\times$, where $I = [n] \smallsetminus \{i\}$. Given

$$\Phi\colon (\rho_i)_*\mathcal{F} = \partial_i(\mathcal{F}) \xrightarrow{\ \sim\ } \mathcal{G}',$$

we extend $\mathcal{G}(\rho_i^\times V) := \mathcal{G}'(V)$ for $V \in \mathcal{P}(S^k_I)$ by

$$\mathcal{G}(U) := \prod_{\substack{\alpha \in U \\ \alpha|_I \not\equiv *}} \mathcal{F}(\{*, \alpha\})$$

otherwise. This is functorial, since $\mathcal{D}$ is pointed, and so Φ exhibits $\mathcal{G}(\rho_i^\times V)$ as the product

$$\mathcal{G}(\rho_i^\times V) = \prod_{\alpha|_I \in V \smallsetminus \{*\}} \mathcal{F}(\{*, \alpha\}).$$

The lifting of Φ itself is then tautological. □

The goal of this section is to prove the following result, which is due to Segal [25], §2, for $k = 1$. Throughout, a lower, resp. upper, d-Segal category means a lower, resp. upper, d-Segal object in Cat, which is not to be confused with a Segal category in the sense of [8].

Theorem 3.4. *Let $k \geq 1$, and $\mathcal{D}$ a pointed category with finite products. The k-dimensional Segal construction $S_\oplus^{\langle k \rangle}(\mathcal{D})$ is a lower $(2k-1)$-Segal category. In particular, it is $2k$-Segal.*

We begin with a proof by direct calculation; a somewhat more conceptual argument will be provided by Theorem 3.6.

Proof. The last part is an application of Proposition 2.10. Now let $n \geq 2k-1$, and set

$$\mathcal{L} = \mathcal{L}([n], 2k-1).$$

For $I, J \in \mathcal{L}$ with $I \supseteq J$, we denote by $\rho_{I,J} \colon S_I^k \to S_J^k$ the corresponding map of pointed sets, and further write $\rho_I = \rho_{[n],I}$ for brevity. We have to show that the canonical functor

$$\operatorname{Sh}(S_n^k, \mathcal{D}) \longrightarrow \varprojlim_{I \in \mathcal{L}} \operatorname{Sh}(S_I^k, \mathcal{D}) \tag{3.1}$$

is an equivalence of categories. Note that by Lemma 3.3, all transition maps on the right-hand side are isofibrations, so that the limit is 1-categorical. Now consider the functor

$$\mathcal{P} \colon \mathcal{L} \longrightarrow \mathrm{Cat}, \ I \longmapsto \mathcal{P}(S_I^k).$$

We form the following version of its Grothendieck construction

$$\pi \colon \mathfrak{X}_{\mathcal{P}} \longrightarrow \mathcal{L}^{\mathrm{op}}.$$

The category $\mathfrak{X}_{\mathcal{P}}$ has objects (I, U), where $I \in \mathcal{L}$ and $U \in \mathcal{P}(S_I^k)$, and there is a unique morphism $(I, U) \leq (J, V)$ if $I \supseteq J$ and $U \subseteq \rho_{I,J}^{\times} V$. The functor π is a cartesian fibration, where a morphism $(I, U) \leq (J, V)$ is cartesian if

$$U = \rho_{I,J}^{\times} V.$$

The category $\varprojlim_{I \in \mathcal{L}} \operatorname{Sh}(S_I^k, \mathcal{D})$ can be identified with the full subcategory of $\operatorname{Fun}(\mathfrak{X}_{\mathcal{P}}^{\mathrm{op}}, \mathcal{D})$ spanned by those presheaves $\mathcal{F}$ which satisfy the following conditions.

(a) The presheaf $\mathcal{F}$ maps cartesian morphisms to isomorphisms in $\mathcal{D}$.

(b) For every $I \in \mathcal{L}$, the restriction of $\mathcal{F}$ to the fibre $\pi^{-1}(I) = \mathcal{P}(S_I^k)$ is a sheaf.

A $\mathcal{D}$-valued presheaf on $\mathcal{P}(S_n^k)$ defines a presheaf on $\mathfrak{X}_{\mathcal{P}}$ via pullback along the functor

$$\varphi_0 \colon \mathfrak{X}_{\mathcal{P}} \longrightarrow \mathcal{P}(S_n^k), \ (I, U) \longmapsto \rho_I^{\times} U.$$

The lower $(2k-1)$-Segal functor (3.1) is then obtained by restricting this pullback along φ_0 to the category of sheaves on $\mathcal{P}(S_n^k)$.

Since the functor φ_0 maps cartesian morphisms to the identity map in $\mathcal{P}(S_n^k)$, it factors over a unique functor $\varphi \colon L\mathfrak{X}_{\mathcal{P}} \to \mathcal{P}(S_n^k)$, where $L\mathfrak{X}_{\mathcal{P}}$ denotes the localization of $\mathfrak{X}_{\mathcal{P}}$ along the set of cartesian morphisms. Note further that imposing condition (a) on a presheaf $\mathcal{F}$ on the category $\mathfrak{X}_{\mathcal{P}}$ is equivalent to the requirement that $\mathcal{F}$ factors through $L\mathfrak{X}_{\mathcal{P}}$.

We obtain an adjunction of presheaf categories as follows,

$$\varphi_! \colon \mathcal{D}_{L\mathfrak{X}_{\mathcal{P}}} \longleftrightarrow \mathcal{D}_{\mathcal{P}(S_n^k)} : \varphi^* \tag{3.2}$$

where the functor φ^* maps the subcategory of sheaves to the subcategory $\varprojlim_{I \in \mathcal{L}} \operatorname{Sh}(S_I^k, \mathcal{D})$. Finally, we introduce the sheafification functor

$$\sigma \colon \mathcal{D}_{\mathcal{P}(S_n^k)} \longrightarrow \operatorname{Sh}(S_n^k, \mathcal{D})$$

as the left adjoint of the inclusion. Here, we need to require the existence of pushouts in $\mathcal{D}$; this assumption is shown to be unnecessary below. Now (3.2) induces an adjunction,

$$\sigma \circ \varphi_! \colon \varprojlim_{I \in \mathcal{L}} \operatorname{Sh}(S_I^k, \mathcal{D}) \longleftrightarrow \operatorname{Sh}(S_n^k, \mathcal{D}) : \varphi^* \tag{3.3}$$

which we claim to be a pair of mutually inverse functors. In order to verify this, we show that the unit and counit are isomorphisms. For the former, it suffices to show that, for every sheaf $\mathcal{G} \in \operatorname{Sh}(S_n^k, \mathcal{D})$ and every subset $\{*, \alpha\} \subseteq S_n^k$ of cardinality 2, the unit morphism

$$(\varphi_! \varphi^* \mathcal{G})(\{*, \alpha\}) \longrightarrow \mathcal{G}(\{*, \alpha\})$$

is invertible. We have

$$(\varphi_! \varphi^* \mathcal{G})(\{*, \alpha\}) \cong \varinjlim_{\substack{(I,U) \in L\mathfrak{X}_{\mathcal{P}}^{\mathrm{op}} \\ \alpha \in \rho_I^{\times}(U)}} \mathcal{G}(\rho_I^{\times}(U)).$$

According to Lemma 3.5 (1) below, the indexing category $\varphi^{\mathrm{op}}/\{*, \alpha\}$ of the colimit has a final object $(I_\alpha, \{*, \alpha|_{I_\alpha}\})$, with $\rho_{I_\alpha}^{\times}(\{*, \alpha|_{I_\alpha}\}) = \{*, \alpha\}$. This immediately implies the claim.

We proceed to prove that, for every object $\mathcal{F} \in \varprojlim_{I\in\mathcal{L}} \mathrm{Sh}(S_I^k, \mathcal{D})$, the counit morphism

$$\mathcal{F} \longrightarrow \varphi^*\sigma\varphi_!\mathcal{F}$$

is invertible. Similarly as above, it suffices to show for all $(J, \{*, \beta\}) \in L\mathfrak{X}_{\mathcal{P}}$ that the map

$$\mathcal{F}(J, \{*, \beta\}) \longrightarrow (\varphi^*\sigma\varphi_!\mathcal{F})(J, \{*, \beta\}) \tag{3.4}$$

is an isomorphism in $\mathcal{D}$. Using Lemma 3.5 (1), we compute the right-hand side as

$$(\sigma\varphi_!\mathcal{F})(\rho_J^\times\{*, \beta\}) \cong \prod_{\alpha\in\rho_J^{-1}(\beta)} (\varphi_!\mathcal{F})(\{*, \alpha\}) \cong \prod_{\alpha\in\rho_J^{-1}(\beta)} \mathcal{F}(I_\alpha, \{*, \alpha|_{I_\alpha}\}).$$

Then Lemma 3.5 (2) implies in particular that the map (3.4) is indeed an isomorphism. □

Lemma 3.5. *In the terminology introduced in the proof of Theorem 3.4, let (J, V) be an object of the category $L\mathfrak{X}_{\mathcal{P}}$. Then the following statements hold.*

(1) *Let $\alpha \in \rho_J^\times(V) \setminus \{*\}$. There is a unique morphism*

$$(I_\alpha, \{*, \alpha|_{I_\alpha}\}) \longrightarrow (J, V)$$

in $L\mathfrak{X}_{\mathcal{P}}$, where

$$I_\alpha = \bigcup_{\alpha_i<\alpha_{i+1}} \{i, i + 1\}.$$

(2) *Let $\mathcal{F}$ be an object of $\varprojlim_{I\in\mathcal{L}} \mathrm{Sh}(S_I^k, \mathcal{D}) \subseteq \mathcal{D}_{L\mathfrak{X}_{\mathcal{P}}}$. There is an isomorphism*

$$\mathcal{F}(J, V) \xrightarrow{\sim} \prod_{\alpha\in\rho_J^\times(V)\setminus\{*\}} \mathcal{F}(I_\alpha, \{*, \alpha|_{I_\alpha}\}),$$

whose components are given by restriction along the unique morphisms from (1).

Proof. The even subsets of $[n]$ of cardinality $2k$ are precisely the disjoint unions of k subsets of the form $\{i, i + 1\}$. Since $\alpha \not\equiv *$, it follows that I_α is a (possibly non-disjoint) union of k such subsets. However, it is contained in the even subset of $[n]$ of cardinality $2k$ obtained by inductively filling for each $\alpha_{i-1} < \alpha_i < \alpha_{i+1}$ either the maximal gap $j < i$ of I_α or its minimal gap $j > i$.

The key observation is that the subsets $I \in \mathcal{L}$ which contain I_α are exactly those with

$$\rho_I^\times(\rho_I(\{*, \alpha\})) = \{*, \alpha\}.$$

This implies that for a morphism in $L\mathfrak{X}_{\mathcal{P}}$ of the form $(I_\alpha, \{*, \alpha|_{I_\alpha}\}) \leftarrow (I, U) \rightarrow (J', V')$, we always have $U = \{*, \alpha|_I\}$. Thus, the only condition on V' is that $\alpha|_{J'} \in V'$, and we can assume without loss of generality that $V = \{*, \alpha|_J\}$. In order to describe morphisms

$$\mu\colon (I_\alpha, \{*, \alpha|_{I_\alpha}\}) \longrightarrow (J, \{*, \alpha|_J\})$$

in $L\mathfrak{X}_{\mathcal{P}}$, we consider α as a sequence of k bars situated in a diagram of $[n]$, signifying the fact that $\alpha_j < \alpha_{j+1}$ by the bar $j|(j + 1)$. An object $(I, \{*, \alpha|_I\})$ corresponds to marking the elements $i \in I \subseteq [n]$, and the zig-zag μ is a sequence of moves which shift the markings. Each move consists of adding and then removing certain markings (adhering to the constraints imposed by the definition of $L\mathfrak{X}_{\mathcal{P}}$).

The object $(I_\alpha, \{*, \alpha|_{I_\alpha}\})$ marks all elements adjacent to a bar; that is, we visualize it as a diagram of the following exemplary form,

$$- - \bullet | \bullet - \bullet | \bullet | \bullet - - \bullet| \;\ldots\; | \bullet - - -$$

where '$\bullet$' indicates a marked element and '$-$' an unmarked element of $[n]$. A single '$\bullet$' at a vertex $i \in [n]$ between two bars (i.e., '$|\bullet|$') corresponds precisely to the case $\alpha_{i-1} < \alpha_i < \alpha_{i+1}$ from above. Since $\alpha|_J \not\equiv *$, this implies that $i \in J$; in fact, this condition states exactly that there is an element of J in every region cut out by the bars.

In order to see that μ is unique (if it exists), we first note that a '$\bullet$' can never cross a bar. Indeed, this would require some move

$$(I, \{*, \alpha|_I\}) \longleftarrow (H, \{*, \alpha|_H\}) \longrightarrow X \in L\mathfrak{X}_{\mathcal{P}}$$

which adds a marking to some $i \in I_\alpha$. Then we can define $\beta \in \mathcal{P}(S^k_H)$ by $\beta|_{H\smallsetminus\{i\}} \equiv \alpha|_{H\smallsetminus\{i\}}$ and by replacing the jump $\alpha_{i-1} = \alpha_i < \alpha_{i+1}$ with $\beta_{i-1} < \beta_i = \beta_{i+1}$; but this contradicts the requirement that the left leg of the move be cartesian.

Then uniqueness follows from the fact that moves which are constrained within different sets of bars commute with one another, while the moves occuring between two particular bars all compose to the same shift of markings.

For the existence of μ, we observe that after adding markings for each '$|\bullet|$' as described above (filling the gaps of I_α; where we can always choose the gap closest to an element of J), we can remove at least one marking adjacent to each bar (with the exception of the '$|\bullet|$', in which case the vertex lies in J already, as we have seen).

Then we can move each '$\bullet$' towards its intended position in J by repeatedly marking the adjacent vertex and removing the original; moreover, once a '$\bullet$' has reached its destination, we can duplicate it. This requires no further sets of the form $\{i, i+1\}$ to cover all markings, that is, we stay within $\mathcal{L}$ in this process (as of course $J \in \mathcal{L}$ itself).

Finally, statement (2) follows from the above, since $\mathcal{F}(J, V) = \mathcal{F}(\rho_J^\times V)$, and similarly,

$$\mathcal{F}(I_\alpha, \{*, \alpha|_{I_\alpha}\}) = \mathcal{F}(\{*, \alpha\});$$

but condition (b) tells us that the restriction $\mathcal{F}|_{\mathcal{P}(S^k_J)}$ to the fibre $\pi^{-1}(J)$ is a sheaf. □

Note that the n-cells of the 1-dimensional Segal construction constitute a pointed category with finite products again. We write $S^{(k)}_\oplus(\mathcal{D}) \in \mathrm{Cat}_{\Delta^{\times k}}$ for the k-fold iterate of $S^{\langle 1\rangle}_\oplus(\mathcal{D})$.

The following result not only provides another perspective on the higher Segal construction, but together with Proposition 2.16, it yields an alternative proof of Theorem 3.4.

Theorem 3.6. *Let $k \geq 1$. The k-dimensional Segal construction of a pointed category $\mathcal{D}$ with finite products is naturally equivalent to the total simplicial object of its k-fold Segal construction,*

$$S^{\langle k\rangle}_\oplus(\mathcal{D}) \xrightarrow{\simeq} TS^{(k)}_\oplus(\mathcal{D}).$$

Proof. For $k = 1$, there is nothing to show. By induction, it suffices to prove, for every $k > 1$,

$$S^{\langle k\rangle}_\oplus(\mathcal{D}) \xrightarrow{\simeq} T(S^{\langle k-1\rangle}_\oplus(S^{\langle 1\rangle}_\oplus(\mathcal{D}))).$$

Therefore, we need to construct, for all $k, n > 0$, a natural equivalence of categories

$$\mathrm{Sh}(S^k_n, \mathcal{D}) \xrightarrow{\simeq} \varprojlim_{\substack{I\cup J=[n]\\ I\leq J}} \mathrm{Sh}(S^{k-1}_I, \mathrm{Sh}(S^1_J, \mathcal{D})). \tag{3.5}$$

Note that by Lemma 3.3, the right-hand side is computed by the 1-categorical limit. Now, we first consider $\mathrm{Sh}(S^{k-1}_I, \mathrm{Sh}(S^1_J, \mathcal{D}))$ as a full subcategory of

$$\begin{aligned} &\mathrm{Fun}(\mathcal{P}(S^{k-1}_I)^{\mathrm{op}}, \mathrm{Fun}(\mathcal{P}(S^1_J)^{\mathrm{op}}, \mathcal{D}))\\ \simeq\ &\mathrm{Fun}(\mathcal{P}(S^{k-1}_I)^{\mathrm{op}} \times \mathcal{P}(S^1_J)^{\mathrm{op}}, \mathcal{D}))\\ \simeq\ &\mathrm{Fun}(\mathcal{P}(S^{k-1}_I \amalg S^1_J)^{\mathrm{op}}, \mathcal{D})),\end{aligned}$$

where $\amalg$ is the coproduct of pointed sets. Define the two maps $\lambda, \rho\colon S^k_n \to S^{k-1}_I \amalg S^1_J$ by

$$\lambda(\alpha) = \begin{cases} \alpha|_I & \text{if } \alpha(I) \subseteq [k-1],\\ * & \text{otherwise,}\end{cases} \quad \text{and } \rho(\alpha) = \begin{cases} \alpha|_J & \text{if } \alpha(J) \subseteq \{k-1, k\},\\ * & \text{otherwise.}\end{cases}$$

Then we claim that an equivalence as in (3.5) descends from the induced functors

$$\mathrm{Sh}(S^k_n, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{P}(S^{k-1}_I \amalg S^1_J)^{\mathrm{op}}, \mathcal{D})),\ \mathcal{F} \longmapsto (W \mapsto \mathcal{F}(\lambda^\times W \cap \rho^\times W)).$$

Firstly, let us show that the essential image of this functor lies in $\mathrm{Sh}(S^{k-1}_I, \mathrm{Sh}(S^1_J, \mathcal{D}))$. This amounts to proving that for every $U \in \mathcal{P}(S^{k-1}_I)$, the assignment $V \mapsto \mathcal{F}(\lambda^\times U \cap \rho^\times V)$ is a sheaf on $\mathcal{P}(S^1_J)$. But this reduces straight-forwardly to the sheaf property of $\mathcal{F}$, as

$$\mathcal{F}(\lambda^\times U \cap \rho^\times V) \xrightarrow{\simeq} \prod_{\varepsilon\in V} \prod_{\alpha\in\lambda^\times U\cap\rho^\times\{*,\varepsilon\}} \mathcal{F}(\{*, \alpha\}) \xleftarrow{\simeq} \prod_{\varepsilon\in V} \mathcal{F}(\lambda^\times U \cap \rho^\times\{*, \varepsilon\}).$$

The analogous argument shows that $U \mapsto \mathcal{F}(\lambda^\times U \cap \rho^\times V)$ is a sheaf for every $V \in \mathcal{P}(S^1_J)$, which yields the rest of the claim, since products of sheaves are computed point-wise.

Next, to see that these functors form into a map into the limit boils down to the transitivity of restriction, $(\alpha|_I)|_{I \setminus J} = \alpha|_{I \setminus J}$ for $I, J \subseteq [n]$ as before.

Finally, to construct the inverse, given $(\mathcal{F}_{I,J}) \in \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Sh}(S^{k-1}_I, \mathrm{Sh}(S^1_J, \mathcal{D}))$, we extend

$$\{*, \alpha\} \longmapsto \mathcal{F}_{I(\alpha),J(\alpha)}(\{*, \alpha|_{I(\alpha)}\})(\{*, \alpha|_{J(\alpha)}\}), \text{ for } I(\alpha) = \alpha^{-1}[k-1],\ J(\alpha) = \{i_\alpha\} \amalg \alpha^{-1}(k),$$

where $i_\alpha \in I(\alpha)$ is the maximal element, to a sheaf on $\mathcal{P}(S^k_n)$. In one direction, we use

$$\lambda^\times\{*, \alpha|_{I(\alpha)}\} \cap \rho^\times\{*, \alpha|_{J(\alpha)}\} = \{*, \alpha\}$$

to see that the two constructions are inverse to one another. Conversely, let $\mathcal{F} \in \mathrm{Sh}(S^k_n, \mathcal{D})$ denote the image of some $(\mathcal{F}_{I,J})$. For every $(U, V) \in \mathcal{P}(S^{k-1}_I) \times \mathcal{P}(S^1_J)$, $I \cap J \neq \emptyset$, we get

$$\mathcal{F}(\lambda^\times U \cap \rho^\times V) \xrightarrow{\ \simeq\ } \mathcal{F}_{I,J}(U)(V)$$

via the following isomorphism,

$$\prod_{\alpha \in \lambda^\times U \cap \rho^\times V} \mathcal{F}_{I(\alpha),J(\alpha)}(\{*, \alpha|_{I(\alpha)}\})(\{*, \alpha|_{J(\alpha)}\}) \xrightarrow{\ \simeq\ } \prod_{(\beta,\varepsilon) \in U \times V} \mathcal{F}_{I,J}(\{*, \beta\})(\{*, \varepsilon\}).$$

This map arises via the mutually inverse identifications $\alpha \mapsto (\lambda(\alpha), \rho(\alpha))$ and $(\beta, \varepsilon) \mapsto \beta \cup \varepsilon$ of the respective indexing sets, with its components given by the fact that

$$(\mathcal{F}_{I,J}) \in \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Sh}(S^{k-1}_I, \mathrm{Sh}(S^1_J, \mathcal{D})),$$

which provides, for each $\alpha = \beta \cup \varepsilon$ as above, a chain of isomorphisms

$$\mathcal{F}_{I(\alpha),J(\alpha)}(\{*, \alpha|_{I(\alpha)}\})(\{*, \alpha|_{J(\alpha)}\}) \cong \mathcal{F}_{I',J'}(\{*, \alpha|_{I'}\})(\{*, \alpha|_{J'}\}) \cong \ldots \cong \mathcal{F}_{I,J}(\{*, \beta\})(\{*, \varepsilon\}),$$

where $I' = I(\alpha) \setminus \{i_\alpha\}$, and $J' = \{i_\alpha - 1\} \amalg J(\alpha)$. □

Definition 3.7. We denote by $\mathcal{E}^\simeq$ the maximal subgroupoid of a category $\mathcal{E}$.

Corollary 3.8 (Delooping)**.** *Let $k \geq 1$, and let $K^\oplus(\mathcal{D}) = \Omega|S^{\langle 1 \rangle}_\oplus(\mathcal{D})^\simeq|$ be the direct sum K-theory space of $\mathcal{D}$. There is a natural homotopy equivalence*

$$\Omega^k|S^{\langle k \rangle}_\oplus(\mathcal{D})^\simeq| \xrightarrow{\ \simeq\ } K^\oplus(\mathcal{D}). \tag{3.6}$$

Proof. By Theorem 3.6 as well as Lemma 3.9 below, this reduces to the case $k = 1$, which is a special case of [25], Proposition 1.5. □

Lemma 3.9. *Let $Y \in \mathrm{Top}_{\Delta^{\times k}}$ be a k-fold simplicial space. Then the natural map $DY \to TY$ induces an equivalence*

$$|Y| \simeq |DY| \xrightarrow{\ \simeq\ } |TY|.$$

Proof. Let us consider $Y_{\bullet,\bullet} \in (\mathrm{Set}_\Delta)_{\Delta^{\times k}}$. When $k = 2$, we obtain the claim from

$$|DY| = |D([m] \mapsto DY_{m,\bullet})| \xleftarrow{\ \simeq\ } |T([m] \mapsto DY_{m,\bullet})| \xrightarrow{\ \simeq\ } |T([m] \mapsto TY_{m,\bullet})| = |TY|, \tag{3.7}$$

where $([m] \mapsto DY_{m,\bullet})$ and $([m] \mapsto TY_{m,\bullet})$ are understood as bisimplicial sets. We obtain the equivalences in (3.7) by [5], Theorem 1.1 (or [26], Theorem 1), with the addition of [5], §7 for the second one. By iterating this argument, the statement follows for all k. □

4. The higher Waldhausen construction

We begin by recalling the following non-additive generalization of a Quillen exact category from [11], Definition 2.4.2. For non-additive examples, see *loc.cit.*, Example 2.4.4 *ff*, as well as Example 5.3.

Definition 4.1. A proto-exact category $\mathcal{E}$ consists of a pointed category $\mathcal{E}$ together with two wide subcategories $\mathcal{E}^{\triangleleft}, \mathcal{E}^{\triangleright} \supseteq \mathcal{E}^{\simeq}$, of so-called admissible monomorphisms, resp. admissible epimorphisms, denoted by $B \rightarrowtail A$, resp. $D \twoheadrightarrow C$, subject to the following conditions. All the maps $0 \rightarrowtail A$ and $A \twoheadrightarrow 0$ are admissible. Moreover, a square of the form

$$\begin{tikzcd} B \arrow[r, tail] \arrow[d, two heads] & A \arrow[d, two heads] \\ C \arrow[r, tail] & D \end{tikzcd} \tag{4.1}$$

is cocartesian if and only if it is cartesian, and a diagram $C \twoheadleftarrow B \rightarrowtail A$, resp. $C \rightarrowtail D \twoheadleftarrow A$, can be completed to a square as in (4.1) by a pushout, resp. pullback.

A short exact sequence is a bicartesian square (4.1) with $C = 0$. A functor $\omega : \mathcal{E} \to \mathcal{B}$ between proto-exact categories is called exact if it preserves short exact sequences.

Now fix a proto-exact category $\mathcal{E}$.

Definition 4.2. A morphism $A \to B$ in $\mathcal{E}$ is called admissible if it factors as the composition

$$\begin{tikzcd} A \arrow[rr] \arrow[dr, two heads] & & B \\ & C \arrow[ur, tail] & \end{tikzcd}$$

of an admissible epimorphism followed by an admissible monomorphism in $\mathcal{E}$.

Consider a sequence of admissible morphisms together with their corresponding (unique up to unique isomorphism, by [4], Lemma 8.4) factorizations as above,

$$\begin{tikzcd} A_k \arrow[rr] \arrow[dr, two heads] & & A_{k-1} \arrow[rr] \arrow[dr, two heads] & & \dots \arrow[rr] \arrow[dr, two heads] & & A_0. \\ & C_k \arrow[ur, tail] & & C_{k-1} \arrow[ur, tail] & \dots & C_1 \arrow[ur, tail] & \end{tikzcd}$$

The sequence $A_k \to A_{k-1} \to \dots \to A_0$ will be called

- acyclic, if $C_{i+1} \rightarrowtail A_i \twoheadrightarrow C_i$ is a short exact sequence in $\mathcal{E}$ for all $0 < i < k$.

An acyclic sequence in $\mathcal{E}$ as above is called

- left exact, if $A_k \to A_{k-1}$ is an admissible monomorphism (equivalently, $A_k \xrightarrow{\simeq} C_k$),
- right exact, if $A_1 \to A_0$ is an admissible epimorphism (i.e., $C_1 \xrightarrow{\simeq} A_0$),
- exact, if it is both left exact and right exact.

Definition 4.3. A proto-abelian category is a pointed category with kernels and cokernels, the collections of which form a proto-exact structure.

Remark 4.4. In [9], Definition 1.2, proto-abelian categories are defined as pointed categories on which the classes of all monomorphisms and epimorphisms define a proto-exact structure. We have changed this terminology slightly, which will prove convenient later, rather than introducing yet another different term. This does not exclude any of the main examples of interest (like Example 5.3 (1)).

Let $k, n \geq 0$. We write $\mathrm{Fun}([k],[n])$ for the category of functors between the standard ordinals $[k]$ and $[n]$, considered as small categories. Note that the objects of this category correspond bijectively to the set of k-simplices of the simplicial set Δ^n.

Definition 4.5. Let $k \geq 0$. For every $n \geq 0$, we define the category

$$S_n^{[k]}(\mathcal{E}) \subseteq \mathrm{Fun}(\mathrm{Fun}([k],[n]), \mathcal{E})$$

to be the full subcategory spanned by all diagrams A satisfying the following conditions.

(a) For every $(k-1)$-simplex α in Δ^n, we have
$$A_{s^*_{k-1}\alpha} = \ldots = A_{s^*_0\alpha} = 0.$$
(b) For every $(k+1)$-simplex γ in Δ^n, the corresponding sequence
$$A_{d^*_{k+1}\gamma} \longrightarrow A_{d^*_k\gamma} \longrightarrow \ldots \longrightarrow A_{d^*_1\gamma} \longrightarrow A_{d^*_0\gamma}$$
is acyclic.

We define $S^{\langle k]}_n(\mathcal{E})$, resp. $S^{[k\rangle}_n(\mathcal{E})$, as the full subcategory of $S^{[k]}_n(\mathcal{E})$ on all A such that

(b') For every $(k+1)$-simplex γ in Δ^n, the following sequence is left, resp. right, exact.
$$A_{d^*_{k+1}\gamma} \longrightarrow A_{d^*_k\gamma} \longrightarrow \ldots \longrightarrow A_{d^*_1\gamma} \longrightarrow A_{d^*_0\gamma}$$

Finally, we introduce $S^{\langle k\rangle}_n(\mathcal{E}) \subseteq S^{[k]}_n(\mathcal{E})$ as the full subcategory of diagrams A which satisfy

(b'') For every $(k+1)$-simplex γ in Δ^n, the sequence
$$A_{d^*_{k+1}\gamma} \longrightarrow A_{d^*_k\gamma} \longrightarrow \ldots \longrightarrow A_{d^*_1\gamma} \longrightarrow A_{d^*_0\gamma}$$
is exact.

By functoriality in $[n]$, we obtain simplicial categories
$$S^{[k]}(\mathcal{E}),\ S^{\langle k]}(\mathcal{E}),\ S^{[k\rangle}(\mathcal{E}),\ S^{\langle k\rangle}(\mathcal{E}) \in \mathrm{Cat}_\Delta\,.$$
We call $S^{\langle k\rangle}(\mathcal{E})$ the k-dimensional Waldhausen construction of $\mathcal{E}$.

Remark 4.6. The $(k+1)$-skeleton of $S^{\langle k\rangle}(\mathcal{E})$ has an immediate description. Namely,
$$S^{\langle k\rangle}_0(\mathcal{E}) \simeq \ldots \simeq S^{\langle k\rangle}_{k-1}(\mathcal{E}) \simeq 0, \text{ and } S^{\langle k\rangle}_k(\mathcal{E}) \simeq \mathcal{E},$$
while $S^{\langle k\rangle}_{k+1}(\mathcal{E})$ is equivalent to the category of k-extensions in $\mathcal{E}$. The dimensionality of the Waldhausen construction also refers to the fact that the k-skeleton of $|S^{\langle k\rangle}(\mathcal{E})^{\simeq}|$ is equivalent to the k-fold suspension $S^k \wedge |\mathcal{E}^{\simeq}|$, rather than the dimension of the diagrams it classifies.

Example 4.7. Let $k \geq 0$, and let $\mathcal{E}$ be a proto-exact category.

(1) For $k = 0$, the degeneracy condition (a) is empty, and therefore,
$$S^{\langle 0\rangle}(\mathcal{E}) \simeq N^{\mathcal{E}}(\mathcal{E}^{\simeq}) \simeq \mathcal{E}$$
is the nerve of the maximal subgroupoid of $\mathcal{E}$, categorified by arbitrary morphisms in $\mathcal{E}$, which is equivalent to the constant object $\mathcal{E}$ itself. Similarly, $S^{\langle 0]}(\mathcal{E}) \simeq N^{\mathcal{E}}(\mathcal{E}^{\triangleleft})$, and dually, $S^{[0\rangle}(\mathcal{E}) \simeq N^{\mathcal{E}}(\mathcal{E}^{\triangleright})$. Rather more subtly, $S^{[0]}(\mathcal{E}) \xrightarrow{\simeq} N^{\mathcal{E}}(\mathcal{E})$ if and only if $\mathcal{E}$ is proto-abelian (in the sense of Definition 4.3), by [13], Proposition 3.1.

(2) For $k = 1$, we recover a version of the original construction $S^{\langle 1\rangle}(\mathcal{E}) \simeq S(\mathcal{E})$ from [29], whose n-cells are given by the category formed by strictly upper triangular diagrams with bicartesian squares, as follows.
$$\begin{tikzcd}
0 \arrow[r, tail] & A_{01} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & A_{02} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & A_{0n} \arrow[d, two heads] \\
& 0 \arrow[r, tail] & A_{12} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \ddots \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & A_{1n} \arrow[d, two heads] \\
& & 0 \arrow[r, tail] & \ddots \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, two heads] \\
& & & 0 \arrow[r, tail] & A_{(n-1)n} \arrow[d, two heads] \\
& & & & 0
\end{tikzcd} \tag{4.2}$$

This is a refinement of Quillen's foundational construction $Q(\mathcal{E})$ in [22], which is the correspondence category on $\mathcal{E}$ with morphisms of the form $C \twoheadleftarrow B \rightarrowtail A$. Namely, the forgetful functor from the edgewise subdivision $ES(\mathcal{E}) \to N^{\mathcal{E}}(Q(\mathcal{E}))$ is an equivalence, that is, the whole diagram $A \in S_{2n+1}(\mathcal{E})$ of shape (4.2) is uniquely recovered from

$$\begin{tikzcd}[column sep=small]
 & A_{(n-1)(n+1)} \arrow[dl, two heads] \arrow[dr, tail] & & \cdots \arrow[dl, two heads] \arrow[dr, tail] & & A_{0(2n)} \arrow[dl, two heads] \arrow[dr, tail] & \\
 & & & \cdots & & & \\
A_{n(n+1)} & & A_{(n-1)(n+2)} & \cdots & A_{1(2n)} & & A_{0(2n+1)}
\end{tikzcd} \tag{4.3}$$

by taking successive pullbacks and pushouts in $\mathcal{E}$.

(3) For $k = 2$, the simplicial category $S^{\langle 2\rangle}(\mathcal{E})$ was introduced by Hesselholt-Madsen [18]. An element $A \in S_4^{\langle 2\rangle}(\mathcal{E})$ of its 4-cells is a diagram of the following form.

$$\begin{tikzcd}[column sep=small]
A_{012} \arrow[rr, tail] & & A_{013} \arrow[rr, tail] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & A_{014} \arrow[d] & & \\
 & & A_{023} \arrow[rr, tail] \arrow[dr, two heads] & & A_{024} \arrow[dr, two heads] \arrow[dd] & & \\
 & & & A_{123} \arrow[rr, tail] & & A_{124} \arrow[dd] & \\
 & & & & A_{034} \arrow[dr, two heads] \arrow[ur, phantom, "\ulcorner"] & & \\
 & & & & & A_{134} \arrow[dr, two heads] & \\
 & & & & & & A_{234}
\end{tikzcd} \tag{4.4}$$

Note that the middle square is neither cartesian nor cocartesian. Rather, the diagram consists of bicartesian cubes (cf. Remark 4.8), as indicated in the following picture.

$$\begin{tikzcd}[column sep=small]
A_{012} \arrow[rr, tail] \arrow[dr, two heads] \arrow[dd] & & A_{013} \arrow[rr, tail] \arrow[dr, two heads] \arrow[dd] & & A_{014} \arrow[dr, two heads] \arrow[dd] & & \\
 & 0 \arrow[rr, tail] \arrow[dd] & & 0 \arrow[rr, tail] \arrow[dd] & & 0 \arrow[dd] & \\
0 \arrow[rr, tail] \arrow[dr, two heads] & & A_{023} \arrow[rr, tail] \arrow[dr, two heads] \arrow[dd] & & A_{024} \arrow[dr, two heads] \arrow[dd] & & \\
 & 0 \arrow[rr, tail] & & A_{123} \arrow[rr, tail] \arrow[dd] & & A_{124} \arrow[dr, two heads] \arrow[dd] & \\
 & & 0 \arrow[rr, tail] \arrow[dr, two heads] & & A_{034} \arrow[dr, two heads] & & 0 \arrow[dd] \\
 & & & 0 \arrow[rr, tail] \arrow[dr, two heads] & & A_{134} \arrow[dr, two heads] & \\
 & & & & 0 \arrow[rr, tail] & & A_{234}
\end{tikzcd} \tag{4.5}$$

Remark 4.8. In general, $S^{\langle k\rangle}(\mathcal{E})$ is composed of $(k+1)$-dimensional bicartesian hypercubes. More precisely, let $A \in S_n^{[k]}(\mathcal{E})$. Then A lies in $S_n^{\langle k]}(\mathcal{E})$ if and only if

$$A_\beta \;=\; \varprojlim_{\substack{\beta<\beta' \\ \beta'-\beta\leq 1}} A_{\beta'} \tag{4.6}$$

for every k-simplex β in Δ^n with $\beta_{k-i} < n-i$ for $0 \leq i \leq k$. Then Lemma 4.9 below implies that the dual condition defines $S_n^{[k\rangle}(\mathcal{E})$ inside $S_n^{[k]}(\mathcal{E})$, so that $A \in S_n^{[k\rangle}(\mathcal{E})$ if and only if

$$\varinjlim_{\substack{\beta'<\beta \\ \beta-\beta'\leq 1}} A_{\beta'} \;=\; A_\beta$$

for every k-simplex β in Δ^n with $i < \beta_i$ for all $0 \le i \le k$. Together, these yield the claim for

$$S_n^{\langle k\rangle}(\mathcal{E}) \simeq S_n^{\langle k]}(\mathcal{E}) \times_{S_n^{[k]}(\mathcal{E})} S_n^{[k\rangle}(\mathcal{E}).$$

In order to see the first statement, note that the sequence of admissible morphisms $A_{d_\bullet^*\gamma}$ corresponding to some $(k+1)$-simplex γ in Δ^n defines a hypercube $\operatorname{conv}(A_{d_\bullet^*\gamma})$, formed by all A_β with $d_{k+1}^*\gamma \le \beta \le d_0^*\gamma$. If the maps (4.6) are isomorphisms, the minimal subhypercubes of $\operatorname{conv}(A_{d_\bullet^*\gamma})$ are cartesian, and hence so is $\operatorname{conv}(A_{d_\bullet^*\gamma})$ as their composition. Therefore,

$$A_{d_{k+1}^*\gamma} \;=\; \varprojlim_{d_{k+1}^*\gamma < \beta \le d_0^*\gamma} A_\beta \;=\; \ker(A_{d_k^*\gamma} \to A_{d_{k-1}^*\gamma}).$$

Conversely, let β be a k-simplex in Δ^n with $\beta_{k-i} < n-i$ for all $0 \le i \le k$. Then we can infer inductively that the hypercube Q_β on all $\beta' \ge \beta$ with $\beta' - \beta \le 1$ is cartesian. First, assume that $|\beta| = \sum \beta_i$ is maximal. Thus, $\beta = (d_0^*)^{n-k}\Delta_n^n - 1 = d_{k+1}^*(d_0^*)^{n-k-1}\Delta_n^n$. But then Q_β is exactly given by the hypercube $\operatorname{conv}(A_{d_\bullet^*(d_0^*)^{n-k-1}\Delta_n^n})$.

In general, consider $\gamma = \beta \amalg \{n\}$. Then $d_{k+1}^*\gamma = \beta \le \beta + 1 \le d_0^*\gamma$, and hence $\operatorname{conv}(A_{d_\bullet^*\gamma})$ contains Q_β entirely. But its complement is covered by hypercubes $Q_{\widetilde{\beta}}$ with $|\widetilde{\beta}| > |\beta|$, which are cartesian by induction. Therefore, so are all of their compositions, and thus so is Q_β.

The following observation makes the inherent symmetry by duality precise.

Lemma 4.9. *The duality on Δ induces equivalences of simplicial categories*

$$S^{\langle k\rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{\langle k\rangle}(\mathcal{E}^{\mathrm{op}}),$$
$$S^{\langle k]}(\mathcal{E}) \xrightarrow{\simeq} S^{[k\rangle}(\mathcal{E}^{\mathrm{op}}),$$
$$S^{[k]}(\mathcal{E}) \xrightarrow{\simeq} S^{[k]}(\mathcal{E}^{\mathrm{op}}).$$

Proof. This is immediate from the definitions. □

Lemma 4.10. *Let $k \ge 0$, $n \ge 1$, $0 \le i \le n$, and let $\mathcal{E}$ be a proto-exact category. The face map*

$$\partial_i\colon S_n^{[k]}(\mathcal{E}) \to S_{n-1}^{[k]}(\mathcal{E})$$

is an isofibration. In particular, the analogous statements hold for $S^{\langle k]}(\mathcal{E})$ and $S^{[k\rangle}(\mathcal{E})$, as well as the higher Waldhausen construction of $\mathcal{E}$ itself.

Proof. Let $\Phi\colon \partial_i(A) \xrightarrow{\sim} B'$ be an isomorphism in $S_{n-1}^{[k]}(\mathcal{E})$. We construct a lift $B \in S_n^{[k]}(\mathcal{E})$ of B' as follows,

$$B\colon ([k] \xrightarrow{\beta} [n]) \longmapsto \begin{cases} A_\beta & \text{if } \beta \notin \operatorname{im}(d_i)_*, \\ B'_\alpha & \text{if } \beta = (d_i)_*\alpha. \end{cases}$$

The map in B for $\beta \le \widetilde{\beta}$ is given by the corresponding arrow in A, resp. B', if $\beta, \widetilde{\beta} \notin \operatorname{im}(d_i)_*$, resp. both $\beta = (d_i)_*\alpha$ and $\widetilde{\beta} = (d_i)_*\widetilde{\alpha}$. Otherwise, we define

$$(B_\beta \to B_{\widetilde{\beta}}) = \begin{cases} (A_\beta \to A_{\widetilde{\beta}} \xrightarrow{\Phi} B'_{\widetilde{\alpha}}) & \text{if } \beta \notin \operatorname{im}(d_i)_* \text{ and } \widetilde{\beta} = (d_i)_*\widetilde{\alpha}, \\ (B'_\alpha \xleftarrow{\Phi} A_\beta \to A_{\widetilde{\beta}}) & \text{if } \beta = (d_i)_*\alpha \text{ and } \widetilde{\beta} \notin \operatorname{im}(d_i)_*. \end{cases}$$

The lifting of Φ itself is then straightforward. □

Remark 4.11. In particular, all the limits with transition maps given by compositions of ∂_i which we consider throughout are computed by the respective 1-categorical limits; we will make no further mention of this henceforth.

This also applies to Theorem 4.15, whose proof in turn can be used to provide an alternative (needlessly complicated) argument for Lemma 4.10 by reducing to the classical case $k = 1$, which further reduces to the case $k = 0$ via the equivalences $S_{n-1}^{\langle 0]}(\mathcal{E}) \simeq S_n^{\langle 1\rangle}(\mathcal{E}) \simeq S_{n-1}^{[0\rangle}(\mathcal{E})$ from Lemma 4.12 below.

Lemma 4.12. *The path spaces of the one-dimensional Waldhausen construction are given by*

$$P^{\triangleleft}S^{\langle 1\rangle}(\mathcal{E}) \simeq S^{\langle 0]}(\mathcal{E}) \simeq N^{\mathcal{E}}(\mathcal{E}^{\triangleleft}), \text{ and dually, } P^{\triangleright}S^{\langle 1\rangle}(\mathcal{E}) \simeq S^{[0\rangle}(\mathcal{E}) \simeq N^{\mathcal{E}}(\mathcal{E}^{\triangleright}).$$

Proof (cf. [11], *Lemma 2.4.9).* Let $n \geq 1$. We construct an inverse to the forgetful functor

$$P^{\triangleright} S^{\langle 1 \rangle}_{n-1}(\mathcal{E}) \longrightarrow S^{[0\rangle}_{n-1}(\mathcal{E}).$$

Given $A \in S^{[0\rangle}_{n-1}(\mathcal{E})$, we define $\widehat{A} \in S^{\langle 1 \rangle}_{n}(\mathcal{E}) \simeq P^{\triangleright} S^{\langle 1 \rangle}_{n-1}(\mathcal{E})$ as its right Kan extension along

$$[n-1] \xrightarrow{d_n} [n] \xrightarrow{-\cup\{n\}} C \hookrightarrow \mathrm{Fun}([1],[n]),$$

where $C = \{\beta \in \mathrm{Fun}([1],[n]) \mid \beta_0 = \beta_1 \text{ or } \beta_1 = n\}$ is considered as a full subcategory. Explicitly, this results in extending $A \cong (A_{0n} \twoheadrightarrow \ldots \twoheadrightarrow A_{(n-1)n})$ by zeroes on the diagonal as in (4.2), then taking successive pullbacks to recover the whole Waldhausen cell. $\square$

We are now prepared to state our main result.

Theorem 4.13. *Let $\mathcal{E}$ be a proto-exact category, and $k \geq 0$. The k-dimensional Waldhausen construction $S^{\langle k \rangle}(\mathcal{E})$ is a $2k$-Segal category.*

Proof. The case $k = 0$ is settled by Example 4.7 (1). For $k = 1$, this is one of the main results of [11], namely Proposition 2.4.8. As observed in *op.cit.*, Example 6.3.3, we can apply the path space criterion to Lemma 4.12 to conclude.

In general, this strategy only works under an additional assumption, as explained in §5. Instead, the result follows from Theorem 4.15 below, together with Proposition 2.19. $\square$

In light of Remark 2.21, another proof for $k = 1$ is provided by Example 4.7 (2), where we have seen that $ES^{\langle 1 \rangle}(\mathcal{E})$ is Segal.

An analogue of Theorem 4.13 in the context of stable ∞-categories is a result of work in progress by Dyckerhoff and Jasso.

Definition 4.14. We write $S^{(k)}(\mathcal{E}) \in \mathrm{Cat}_{\Delta^{\times k}}$ for the k-fold iterate of the 1-dimensional Waldhausen construction, where each $S^{\langle 1 \rangle}_n(\mathcal{E})$ carries the point-wise proto-exact structure.

Theorem 4.15. *Let $k \geq 0$, and let $\mathcal{E}$ be a proto-exact category. There is a natural equivalence of simplicial categories between the k-dimensional Waldhausen construction of $\mathcal{E}$ and the total simplicial object of its k-fold Waldhausen construction,*

$$S^{\langle k \rangle}(\mathcal{E}) \xrightarrow{\simeq} TS^{(k)}(\mathcal{E}).$$

Proof. For $k \leq 1$, this is tautological. By the same reasoning as in the proof of Theorem 3.6, it is sufficient to construct, for all $n, k \geq 1$, a natural equivalence of categories

$$S^{\langle k \rangle}_n(\mathcal{E}) \xrightarrow{\simeq} \varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} S^{\langle k-1 \rangle}_I (S^{\langle 1 \rangle}_J(\mathcal{E})).$$

To define the functor, we use that the right-hand side is a full subcategory of

$$\begin{aligned} &\varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Fun}(\mathrm{Fun}([k-1], I), \mathrm{Fun}(\mathrm{Fun}([1], J), \mathcal{E})) \\ \simeq\ &\varprojlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Fun}(\mathrm{Fun}([k-1], I) \times \mathrm{Fun}([1], J), \mathcal{E}) \\ \simeq\ &\mathrm{Fun}(\varinjlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Fun}([k-1], I) \times \mathrm{Fun}([1], J), \mathcal{E}). \end{aligned}$$

Consider the following full subcategory of the indexing category for its elements,

$$\{(\alpha, \varepsilon) \in \varinjlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Fun}([k-1], I) \times \mathrm{Fun}([1], J) \mid \alpha_{k-1} = \varepsilon_0\}.$$

This is equivalent to $\mathrm{Fun}([k],[n])$ via $(\alpha, \varepsilon) \mapsto \alpha \cup \varepsilon$, with inverse induced by the functor

$$\mathrm{Fun}([k],[n]) \longrightarrow \varinjlim_{\substack{I \cup J = [n] \\ I \leq J}} \mathrm{Fun}([k-1], I) \times \mathrm{Fun}([1], J), \ \beta \longmapsto (\beta|_{[k-1]}, \beta|_{\{k-1,k\}}).$$

Now let $A \in S_n^{\langle k\rangle}(\mathcal{E})$. Then its left Kan extension $A^!$ along the inclusion

$$\operatorname{Fun}([k],[n]) \hookrightarrow \{(\alpha,\varepsilon) \in \varinjlim_{\substack{I\cup J=[n]\\ I\leq J}} \operatorname{Fun}([k-1],I)\times \operatorname{Fun}([1],J) \mid \alpha_{k-1}=\varepsilon_0 \text{ or } \varepsilon_0=\varepsilon_1\}$$

amounts to an iterated extension by zeroes, as in the proof of Lemma 4.12. Then we define the image $\widehat{A} \in \varprojlim_{\substack{I\cup J=[n]\\ I\leq J}} S_I^{\langle k-1\rangle}(S_J^{\langle 1\rangle}(\mathcal{E}))$ of A as a further left Kan extension to the whole indexing category $\varinjlim_{\substack{I\cup J=[n]\\ I\leq J}} \operatorname{Fun}([k-1],I)\times\operatorname{Fun}([1],J)$, which again is an iterated version of the corresponding construction in the proof of Lemma 4.12. Note that the Kan extension exists, since it only involves taking cokernels of admissible monomorphisms.

The construction is fully faithful by universality, and essential surjectivity is seen as follows. For $A' \in \varprojlim_{\substack{I\cup J=[n]\\ I\leq J}} S_I^{\langle k-1\rangle}(S_J^{\langle 1\rangle}(\mathcal{E}))$, a preimage $A \in S_n^{\langle k\rangle}(\mathcal{E})$ is given by glueing together the sequence $A_{d_{k+1}^*\gamma} \to \ldots \to A_{d_0^*\gamma}$ as $A'_{(\gamma_0\cdots\gamma_{k-1},\gamma_{k-1}\gamma_k\gamma_{k+1})} \times_{A'_{(\gamma_0\cdots\gamma_{k-1},\gamma_k\gamma_{k+1})}} A'_{(\gamma_0\cdots\gamma_k,\gamma_k\gamma_{k+1})}$, which is meant to be short-hand notation for the following sequence,

$$A'_{(\gamma_0\cdots\gamma_{k-1},\gamma_{k-1}\gamma_k)} \to A'_{(\gamma_0\cdots\gamma_{k-1},\gamma_{k-1}\gamma_{k+1})} \to A'_{(\gamma_0\cdots\gamma_{k-2}\gamma_k,\gamma_k\gamma_{k+1})} \to \ldots \to A'_{(\gamma_1\cdots\gamma_k,\gamma_k\gamma_{k+1})}.$$

$\square$

Example 4.16. As a special case of Theorem 4.15, we illustrate the equivalence of categories $S_4^{\langle 2\rangle}(\mathcal{E}) \simeq S^{(2)}_{\{0,1\},\{1,2,3,4\}}(\mathcal{E}) \times_{S^{(2)}_{\{0,1\},\{2,3,4\}}(\mathcal{E})} S^{(2)}_{\{0,1,2\},\{2,3,4\}}(\mathcal{E}) \times_{S^{(2)}_{\{0,1,2\},\{3,4\}}(\mathcal{E})} S^{(2)}_{\{0,1,2,3\},\{3,4\}}(\mathcal{E})$ in the following diagram, where the glueing is represented by the two dashed sequences.

$$\begin{tikzcd}
 & & & & & & A_{123} \arrow[drr, tail] & & & & & & \\
 & & & & A_{023} \arrow[urr, two heads] \arrow[drr, tail] & & & & A_{124} \arrow[dr] & & & & \\
 & & & \bullet \arrow[ur] \arrow[drr, dashed] & & & A_{024} \arrow[urr, two heads] \arrow[dr] & & & \bullet \arrow[ddr] & & & \\
 & & & & & \bullet \arrow[ur] \arrow[dr, dashed] & & \bullet \arrow[dr] \arrow[urr, dashed] & & & & & \\
 & & A_{013} \arrow[rr, tail] \arrow[uur] & & A_{014} \arrow[ur] \arrow[rr] & & \bullet \arrow[rr] \arrow[ur, dashed] & & A_{034} \arrow[rr, two heads] \arrow[drrrr, two heads] & & A_{134} \arrow[drr, two heads] & & \\
A_{012} \arrow[uurr, tail] \arrow[urrrr, tail] & & & & & & & & & & & & A_{234}
\end{tikzcd}$$

Let $k \geq 1$. As can be seen either directly or with the help of Theorem 4.15, the functor

$$\Phi\colon \mathrm{ExCat} \longrightarrow \mathrm{Top}_*,\ \mathcal{E} \longmapsto |S^{\langle k-1\rangle}(\mathcal{E})^{\simeq}|, \tag{4.7}$$

satisfies the hypotheses of [17], §1.3. This permits us to draw the following consequences on geometric realizations.

Corollary 4.17 (Additivity)**.** *Let $\mathcal{E}$ be an exact category, and $k \geq 1$. Then the map*

$$|S^{\langle k\rangle}(S_2(\mathcal{E}))^{\simeq}| \xrightarrow{\simeq} |S^{\langle k\rangle}(\mathcal{E})^{\simeq}| \times |S^{\langle k\rangle}(\mathcal{E})^{\simeq}|$$

induced by the functor $(\partial_2,\partial_0)\colon S_2(\mathcal{E}) \to \mathcal{E}\times\mathcal{E}$ is a weak equivalence. In fact, the simplicial space $|S^{\langle k\rangle}(S_\bullet(\mathcal{E}))^{\simeq}|$ is a Segal space.

Proof. By Theorem 4.15, this is precisely [17], Theorem 1.3.5 (2), with Φ as in (4.7). $\square$

Needless to say, the other versions of additivity in [17], Theorem 1.3.5, hold as well; this also allows us to deduce the following as in *loc.cit.* via the proof of [20], Proposition 3.6.2.

Corollary 4.18 (Delooping). *Let $k \geq 1$, and let $K(\mathcal{E})$ denote the algebraic K-theory space of the exact category $\mathcal{E}$. There is a natural homotopy equivalence*

$$\Omega^k |S^{\langle k\rangle}(\mathcal{E})^{\simeq}| \xrightarrow{\simeq} K(\mathcal{E}).$$

Remark 4.19. Similarly to Corollary 3.8, we can also use Theorem 4.15 to immediately reduce to the case $k = 1$ ([29], Theorem 1.4.2 and Proposition 1.3.2, resp. Proposition 1.5.3).

In conclusion, the algebraic K-theory spectrum of $\mathcal{E}$ is given by the sequence of maps

$$|S^{\langle 0\rangle}(\mathcal{E})^{\simeq}| \longrightarrow \Omega|S^{\langle 1\rangle}(\mathcal{E})^{\simeq}| \longrightarrow \Omega^2|S^{\langle 2\rangle}(\mathcal{E})^{\simeq}| \longrightarrow \dots,$$

where, for each $k \geq 0$, the morphism

$$|S^{\langle k\rangle}(\mathcal{E})^{\simeq}| \longrightarrow \Omega|S^{\langle k+1\rangle}(\mathcal{E})^{\simeq}|$$

is induced by the inclusions of the cells $\theta_n \colon S_n^{\langle k\rangle}(\mathcal{E}) \hookrightarrow S_{n+1}^{\langle k+1\rangle}(\mathcal{E})$ as $S_1^{\langle 1\rangle}(S_n^{\langle k\rangle}(\mathcal{E}))$, extended by zeroes appropriately, for all $n \geq 0$. Namely, [29], Lemma 1.5.2, constructs the map via

$$S^{(k)}(\mathcal{E}) \longrightarrow P^{\triangleleft}S^{(k+1)}(\mathcal{E}) \longrightarrow S^{(k+1)}(\mathcal{E}), \tag{4.8}$$

inducing a map $S^{(k)}(\mathcal{E}) \to L^{\triangleleft}S^{(k+1)}(\mathcal{E})$ to the simplicial loop space. Totalizing (4.8) yields

$$S^{\langle k\rangle}(\mathcal{E}) \xrightarrow{\theta} P^{\triangleleft}S^{\langle k+1\rangle}(\mathcal{E}) \longrightarrow S^{\langle k+1\rangle}(\mathcal{E})$$

by Lemma 2.18. As before, this produces the desired map $S^{\langle k\rangle}(\mathcal{E}) \longrightarrow L^{\triangleleft}S^{\langle k+1\rangle}(\mathcal{E})$.

Finally, the following result can be understood as a refinement of the delooping theorem in that morally it states that $S^{\langle k\rangle}(\mathcal{E})$ exhibits the E_k-algebra structure on $K(\mathcal{E})$.

Corollary 4.20. *Let $\mathcal{E}$ be an exact category, and $k \geq 1$. The simplicial space $K(S^{\langle k\rangle}(\mathcal{E}))$ is a lower $(2k-1)$-Segal space.*

Proof. From Theorem 4.15 as well as Lemma 4.21 below, we deduce that

$$K(S^{\langle k\rangle}(\mathcal{E})) \simeq K(TS^{(k)}(\mathcal{E})) \simeq TK(S^{(k)}(\mathcal{E})).$$

But by the additivity theorem, $K(S^{(k)}(\mathcal{E}))$ is a Segal space in every direction, and therefore, the statement follows from Proposition 2.16. □

Lemma 4.21. *Let $\mathcal{E}, \mathcal{B}, \mathcal{D}$ be proto-exact categories, let $\omega \colon \mathcal{E} \to \mathcal{B}$ be an exact isofibration, and let $\nu \colon \mathcal{D} \to \mathcal{B}$ be an exact functor. Then*

$$S(\mathcal{E} \times_{\mathcal{B}} \mathcal{D}) \xrightarrow{\simeq} S(\mathcal{E}) \times_{S(\mathcal{B})} S(\mathcal{D}),$$

and the right-hand side agrees with the 1-categorical fibre product. In particular,

$$K(\mathcal{E} \times_{\mathcal{B}} \mathcal{D}) \xrightarrow{\simeq} K(\mathcal{E}) \times_{K(\mathcal{B})} K(\mathcal{D}).$$

Proof. Firstly, the functor $S_n(\omega)$ is an isofibration, for all $n \in \mathbb{N}$. Indeed, this can be proven inductively (cf. [29], Lemma 1.6.6). The case $n = 1$ is our assumption. Given an isomorphism

$$\begin{tikzcd}
\omega(M_0) \arrow[r, hook] \arrow[d, "\sim"'] & \cdots \arrow[r, hook] & \omega(M_{n-1}) \arrow[r, hook] \arrow[d, "\sim"'] & \omega(M_n) \arrow[d, "\sim"', "f"] \\
V_0 \arrow[r, hook] & \cdots \arrow[r, hook] & V_{n-1} \arrow[r, hook] & V_n,
\end{tikzcd}$$

in $S_{n+1}(\mathcal{B}) \simeq S_n^{\langle 0\rangle}(\mathcal{B})$, for $n \geq 1$, we can lift all but the last vertical arrow f, and subsequently take the pushout

$$\begin{tikzcd}
M_0 \arrow[r, hook] \arrow[d, "\sim"'] & \cdots \arrow[r, hook] & M_{n-1} \arrow[r, hook] \arrow[d, "\sim"'] \arrow[dr, phantom, "\ulcorner"] & M_n \arrow[d, two heads, "\varphi"] \\
N_0 \arrow[r, hook] & \cdots \arrow[r, hook] & N_{n-1} \arrow[r, hook] & N.
\end{tikzcd}$$

Then φ is a lift of f. This shows the second part of the first claim. Note that we use repeatedly that all functors involved are exact. Now, applying $S_{n+1}(-)$ to the projections yields a unique functor $S_{n+1}(\mathcal{E} \times_{\mathcal{B}} \mathcal{D}) \to S_{n+1}(\mathcal{E}) \times_{S_{n+1}(\mathcal{B})} S_{n+1}(\mathcal{D})$. But under Lemma 4.12 again, this becomes the equivalence

$$S_n^{\langle 0]}(\mathcal{E} \times_{\mathcal{B}} \mathcal{D}) = N_n^{\mathcal{E} \times_{\mathcal{B}} \mathcal{D}}((\mathcal{E} \times_{\mathcal{B}} \mathcal{D})^{\triangleleft}) \simeq N_n^{\mathcal{E}}(\mathcal{E}^{\triangleleft}) \times_{N_n^{\mathcal{B}}(\mathcal{B}^{\triangleleft})} N_n^{\mathcal{D}}(\mathcal{D}^{\triangleleft}) = S_n^{\langle 0]}(\mathcal{E}) \times_{S_n^{\langle 0]}(\mathcal{B})} S_n^{\langle 0]}(\mathcal{D})$$

since by definition, $(\mathcal{E} \times_{\mathcal{B}} \mathcal{D})^{\triangleleft} = \mathcal{E}^{\triangleleft} \times_{\mathcal{B}^{\triangleleft}} \mathcal{D}^{\triangleleft}$, and because

$$N_n(\mathcal{E} \times_{\mathcal{B}} \mathcal{D}) = \mathrm{Fun}([n], \mathcal{E} \times_{\mathcal{B}} \mathcal{D}) \simeq \mathrm{Fun}([n], \mathcal{E}) \times_{\mathrm{Fun}([n],\mathcal{B})} \mathrm{Fun}([n], \mathcal{D}) = N_n(\mathcal{E}) \times_{N_n(\mathcal{B})} N_n(\mathcal{D}).$$

Finally, for the last claim, $(-)^{\simeq}$ and Ω are both right adjoints, while geometric realization commutes with fibre products by [7]. $\square$

5. Stringent categories and path spaces

Let $\mathcal{E}$ be a proto-exact category. In this section, we investigate which further higher Segal conditions the higher dimensional Waldhausen construction of $\mathcal{E}$ and its variants satisfy under the following additional homological algebraic assumption on $\mathcal{E}$.

Definition 5.1. A stringent category is a proto-exact category $\mathcal{E}$ such that for every triangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "g\circ f"'] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

of admissible morphisms in $\mathcal{E}$, the induced sequence of maps

$$\ker(f) \longrightarrow \ker(g \circ f) \longrightarrow \ker(g) \longrightarrow \mathrm{coker}(f) \longrightarrow \mathrm{coker}(g \circ f) \longrightarrow \mathrm{coker}(g)$$

is exact.

Remark 5.2. A proto-exact category $\mathcal{E}$ is stringent if and only if the functor

$$P^{\triangleleft} P^{\triangleright} S^{\langle 2 \rangle}(\mathcal{E}) \longrightarrow S^{[0]}(\mathcal{E}),\ A \longmapsto A_{[0]\oplus - \oplus[0]},$$

defines an equivalence of simplicial categories. Indeed, the equivalence

$$P^{\triangleleft} P^{\triangleright} S_2^{\langle 2 \rangle}(\mathcal{E}) \xrightarrow{\simeq} S_2^{[0]}(\mathcal{E})$$

is precisely the definition of stringency, while the converse is part of Proposition 5.11.

Example 5.3. By symmetry of the definition, $\mathcal{E}$ is stringent if and only if $\mathcal{E}^{\mathrm{op}}$ is. If $\mathcal{E}$ is stringent, then so is $\mathrm{Fun}(Q, \mathcal{E})$, for a small category Q.

(1) In particular, the category $\mathrm{Fun}(Q, \mathrm{vect}_{\mathbb{F}_1})$ of representations of any small category Q in (finite) $\mathbb{F}_1$-vector spaces is stringent, where $\mathrm{vect}_{\mathbb{F}_1}$ is the wide subcategory of $\mathrm{Fin}_{\times}$ on all $f\colon V \to W$ which are injective outside of the kernel $f^{-1}(*)$.

(2) Consider the pointed category $\mathcal{E}$ on a non-zero object V with $\mathrm{End}(V) = \{0, 1, \varepsilon\}$, such that $\varepsilon^2 = 0$. It can easily be verified directly that ε admits neither a kernel nor a cokernel, and thus $\mathcal{E}$ is stringent.

If F is a field, then the F-linear Cauchy completion of $\mathcal{E}$ is an additive stringent category, namely the category of finite free $F[x]/(x^2)$-modules.

Remark 5.4. An additive category is stringent if and only if it is weakly idempotent complete in the sense of [4], Definition 7.2, or equivalently, any of the characterizations given there; in particular, the cancellation axiom *loc.cit.*, Corollary 7.7, suffices to recover the stringency property.

This is one of Heller's axioms (cf. *op.cit.*, Proposition B.1), the rest of which also follow from weak idempotent completeness. In our non-additive setting, the entirety of these axioms (with the obvious exception of additivity) still provides an equivalent characterization of stringency. Indeed, the proof of *op.cit.*, Proposition 8.11, does not require additivity, and the converse is shown in *loc.cit.*, Lemma 8.12, as well as Proposition 5.5 below.

As a consequence, the snake lemma holds in any stringent category $\mathcal{E}$. The neat argument presented in *loc.cit.*, Corollary 8.13, does not quite apply here (as it ultimately relies on the additive structure of an exact category); however, the proof of [16], Proposition 4.3, does.

Proposition 5.5. *Let $\mathcal{E}$ be a stringent category, and consider a diagram of the form*

$$\begin{tikzcd} A' \arrow[r, tail] \arrow[d] & B' \arrow[r, two heads] \arrow[d, tail] & C' \arrow[d, tail] \\ A \arrow[r, tail] \arrow[d] & B \arrow[r, two heads] \arrow[d, two heads] & C \arrow[d, two heads] \\ A'' \arrow[r, tail] & B'' \arrow[r, two heads] & C'' \end{tikzcd}$$

in $\mathcal{E}$, where all rows as well as all columns but the first are exact. Then $A' \rightarrowtail A \twoheadrightarrow A''$ is a short exact sequence as well.

Proof. The kernel-cokernel sequence for the lower triangle (of admissible maps) in the square

$$\begin{tikzcd} B' \arrow[r, two heads] \arrow[d, tail] \arrow[dr] & C' \arrow[d, tail] \\ B \arrow[r, two heads] & C \end{tikzcd}$$

is given by $0 \longrightarrow A' \longrightarrow A \longrightarrow B'' \longrightarrow C'' \longrightarrow 0$, and $\ker(B'' \to C'') = A''$. ☐

We are now prepared to state the first main result of this section.

Proposition 5.6. *Suppose $\mathcal{E}$ is a stringent category. Then the simplicial categories $S^{\langle k]}(\mathcal{E})$ and $S^{[k\rangle}(\mathcal{E})$ are lower $(2k-1)$-Segal.*

Remark 5.7. When $k \geq 2$, the assumption in Proposition 5.6 that $\mathcal{E}$ be stringent is necessary, which is illustrated by the following observation, at least in the additive case. Suppose $\mathcal{E}$ is not weakly idempotent complete. Then $S^{\langle 1]}(\mathcal{E})$ is not lower 3-Segal.

Indeed, there exist $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{E}$ such that g and $g \circ f$ are admissible monomorphisms but f is not. However, by [4], Remark 8.2, it can be written as a composition of strict maps

$$A \xrightarrow{(1,f)} A \oplus B \xrightarrow{(0,1)} B.$$

Now suppose f admits a kernel D and consider the following possible element of $S_4^{\langle 1]}(\mathcal{E})$.

$$\begin{tikzcd} 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "{(1,f)}"] \\ & 0 \arrow[r, tail] & A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \oplus B \arrow[d, "{(0,1)}"] \\ & & 0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "g"] \\ & & & 0 \arrow[r, tail] & C \arrow[d] \\ & & & & 0 \end{tikzcd} \tag{5.1}$$

Then the triple of diagrams

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D \arrow[d] \\
0 \arrow[r, tail] & A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d] \\
 & 0 \arrow[r, tail] & 0
\end{tikzcd}
\qquad
\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "{(1,f)}"] \\
0 \arrow[r, tail] & A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \oplus B \arrow[d, "{(0,g)}"] \\
 & 0 \arrow[r, tail] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \oplus B \arrow[d, "{(0,1)}"] \\
0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "g"] \\
 & 0 \arrow[r, tail] & C
\end{tikzcd}$$

defines an element in the right-hand side of the lower 3-Segal map for $S_4^{\langle 1]}(\mathcal{E})$. However, it does not lie in its essential image, because the sequence

$$D \rightarrowtail A \xrightarrow{f} B$$

indexed by $\{0, 2, 4\}$ is not left exact (the map f not being strict), so (5.1) $\notin S_4^{\langle 1]}(\mathcal{E})$.

Similarly, the dual argument shows that the 4-cells of $S^{[1\rangle}(\mathcal{E})$ also do not satisfy the lower 3-Segal condition (by Lemma 4.9).

Example 5.8. Let us illustrate the lowest 3-Segal conditions for $S^{\langle 2\rangle}(\mathcal{E})$, which is more conveniently done by depicting an element of its 4-cells as the following projection of (4.4).

$$\begin{tikzcd}
 & & & & A_{123} \arrow[dr, tail] & & & \\
 & & & A_{023} \arrow[ur, two heads] \arrow[dr, dashed, tail] & & A_{124} \arrow[dr, two heads] & & \\
 & & & & A_{024} \arrow[ur, dashed] \arrow[dr, dashed] & & & \\
 & & A_{013} \arrow[r, tail] \arrow[uur] & A_{014} \arrow[ur, dashed] \arrow[rr] & & A_{034} \arrow[r, two heads] \arrow[drr, dashed, two heads] & A_{134} \arrow[dr, two heads] & \\
 & A_{012} \arrow[ur, tail] \arrow[urr, dashed, tail] & & & & & & A_{234}
\end{tikzcd} \tag{5.2}$$

The dashed part marks the image of (5.2) in the right-hand side of the upper 3-Segal map

$$S_4^{\langle 2\rangle}(\mathcal{E}) \longrightarrow S_3^{\langle 2\rangle}(\mathcal{E}) \times_{S_2^{\langle 2\rangle}(\mathcal{E})} S_3^{\langle 2\rangle}(\mathcal{E}).$$

The upper 3-Segal condition says that the whole diagram (5.2) is uniquely recovered from the dashed subdiagram. Note that the complementary statement (for the lower 3-Segal map) is false in general. In fact, it is equivalent to uniquely filling the frame of short exact sequences

$$\begin{tikzcd}
C_4 \arrow[r, tail] \arrow[d, tail] & A_{023} \arrow[r, two heads] \arrow[d, dashed, tail] & A_{123} \arrow[d, tail] \\
C_3 \arrow[r, dashed, tail] \arrow[d, two heads] & A'_{024} \arrow[r, dashed, two heads] \arrow[d, dashed, two heads] & A_{124} \arrow[d, two heads] \\
C_2 \arrow[r, tail] & C_1 \arrow[r, two heads] & C_0
\end{tikzcd} \tag{5.3}$$

where $C_i = \operatorname{coker}(A_{d_3^* d_i^* \Delta_4^4} \rightarrowtail A_{d_2^* d_i^* \Delta_4^4}) \cong \ker(A_{d_1^* d_i^* \Delta_4^4} \twoheadrightarrow A_{d_0^* d_i^* \Delta_4^4})$. However, there is an obstruction to this, which is parametrized by the quotient groupoid

$$[\operatorname{Ext}^1(C_0, C_4)/\operatorname{Hom}(C_0, C_4)],$$

as calculated in [9], Lemma 2.30 and Proposition 2.38, assuming that $\mathcal{E}$ is abelian.

Our next observation will prove essential for our inductive arguments.

Proposition 5.9 (Hyperplane lemma)**.** *Let $1 \leq k \leq l < m \leq n$, and let $\mathcal{E}$ be a stringent category. Then there is a natural functor*

$$\eta_{lm}^{\triangleleft} \colon S_n^{\langle k]}(\mathcal{E}) \longrightarrow S_l^{\langle k-1]}(\mathcal{E}),\ A \longmapsto (\beta \mapsto \operatorname{coker}(A_{\beta \cup \{l\}} \rightarrowtail A_{\beta \cup \{m\}})).$$

Dually, there is a corresponding natural functor

$$\eta^{\rhd}_{lm}\colon S^{[k\rangle}_n(\mathcal{E}) \longrightarrow S^{[k-1\rangle}_l(\mathcal{E}),\ A \longmapsto (\beta \mapsto \ker(A_{\{n-m\}\cup\beta} \twoheadrightarrow A_{\{n-l\}\cup\beta})).$$

Moreover, both of these restrict to functors on the higher Waldhausen construction,

$$S^{\langle k\rangle}_n(\mathcal{E}) \underset{\eta^{\rhd}_{lm}}{\overset{\eta^{\lhd}_{lm}}{\rightrightarrows}} S^{\langle k-1\rangle}_l(\mathcal{E}).$$

Proof. Let γ be a k-simplex in Δ^l, and $\gamma' = \gamma \amalg \{m\}$. If $l \in \gamma$, then the sequence $\eta^{\lhd}_{lm}(A)_{d^*_\bullet\gamma}$ is given by

$$\operatorname{coker}(A_{d^*_{k+1}\gamma'} \rightarrowtail A_{d^*_k\gamma'}) \rightarrowtail A_{d^*_{k-1}\gamma'} \longrightarrow A_{d^*_{k-2}\gamma'} \longrightarrow \ldots \longrightarrow A_{d^*_0\gamma'}$$

which of course is indeed left exact. Furthermore, it is exact if and only if A lies in $S^{\langle k\rangle}_n(\mathcal{E})$, by definition.

Now assume that $l \notin \gamma$. Then, for each vertex $0 < i < k$, let us write

$$\begin{tikzcd}
(A_{d^*_{i+1}\gamma\cup\{l\}} \twoheadrightarrow)\ B_{i+1} \arrow[r, tail] \arrow[d, dashed] & A_{d^*_i\gamma\cup\{l\}} \arrow[r, two heads] \arrow[d, dashed] & B_i\ (\rightarrowtail A_{d^*_{i-1}\gamma\cup\{l\}}) \arrow[d, dashed] \\
(A_{d^*_{i+1}\gamma\cup\{m\}} \twoheadrightarrow)\ C_{i+1} \arrow[r, tail] & A_{d^*_i\gamma\cup\{m\}} \arrow[r, two heads] & C_i\ (\rightarrowtail A_{d^*_{i-1}\gamma\cup\{m\}})
\end{tikzcd}$$

for the corresponding short exact sequence. Taking cokernels yields a diagram as follows.

$$\begin{tikzcd}
B_{i+1} \arrow[r, tail] \arrow[d, tail] & C_{i+1} \arrow[r, two heads] \arrow[d, tail] & D_{i+1} \arrow[d] \\
A_{d^*_i\gamma\cup\{l\}} \arrow[r, tail] \arrow[d, two heads] & A_{d^*_i\gamma\cup\{m\}} \arrow[r, two heads] \arrow[d, two heads] & \eta^{\lhd}_{lm}(A)_{d^*_i\gamma} \arrow[d] \\
B_i \arrow[r, tail] & C_i \arrow[r, two heads] & D_i
\end{tikzcd}$$

By the snake lemma, the right vertical sequence is short exact. Note that if $A \in S^{\langle k\rangle}_n(\mathcal{E})$,

$$B_1 \xrightarrow{\sim} A_{d^*_0\gamma\cup\{l\}} \text{ and } C_1 \xrightarrow{\sim} A_{d^*_0\gamma\cup\{m\}}$$

which immediately implies also $D_1 \xrightarrow{\sim} \eta^{\lhd}_{lm}(A)_{d^*_0\gamma}$ by definition. It remains to prove that

$$\eta^{\lhd}_{lm}(A)_{d^*_k\gamma} \longrightarrow \eta^{\lhd}_{lm}(A)_{d^*_{k-1}\gamma}$$

is an admissible monomorphism. In order to see this, we may show that the diagram

$$\begin{tikzcd}
A_{d^*_k\gamma\cup\{l\}} \arrow[r, tail] \arrow[d] & A_{d^*_k\gamma\cup\{m\}} \arrow[d] \\
A_{d^*_{k-1}\gamma\cup\{l\}} \arrow[r, tail] \arrow[d] & A_{d^*_{k-1}\gamma\cup\{m\}} \arrow[d] \\
0 \arrow[r, tail] & A_{d^*_{k-1}d^*_k\gamma\cup\{m-1,m\}}
\end{tikzcd} \tag{5.4}$$

is cartesian, by Lemma 5.10 below. In fact, we claim that it is the composition of pullbacks

$$\begin{tikzcd}
A_{d^*_k\gamma\cup\{l\}} \arrow[r, tail] \arrow[d] & A_{d^*_k\gamma\cup\{l+1\}} \arrow[r, tail] \arrow[d] & \ldots \arrow[r, tail] & A_{d^*_k\gamma\cup\{m\}} \arrow[d] \\
A_{d^*_{k-1}\gamma\cup\{l\}} \arrow[r, tail] \arrow[d] & A_{d^*_{k-1}\gamma\cup\{l+1\}} \arrow[r, tail] \arrow[d] & \ldots \arrow[r, tail] & A_{d^*_{k-1}\gamma\cup\{m\}} \arrow[d] \\
0 \arrow[r, tail] & A_{d^*_{k-1}d^*_k\gamma\cup\{l,l+1\}} \arrow[r, tail] & \ldots \arrow[r, tail] & A_{d^*_{k-1}d^*_k\gamma\cup\{m-1,m\}}.
\end{tikzcd}$$

To prove this claim, for each $l \leq j < m$, we have the diagram

$$\begin{tikzcd} A_{d_k^*\gamma\cup\{j\}} \arrow[r, tail] \arrow[d] & A_{d_k^*\gamma\cup\{j+1\}} \arrow[d] \\ A_{d_{k-1}^*\gamma\cup\{j\}} \arrow[r, tail] \arrow[d] & A_{d_{k-1}^*\gamma\cup\{j+1\}} \arrow[d] \\ 0 \arrow[r, tail] & A_{d_{k-1}^* d_k^*\gamma\cup\{j,j+1\}} \end{tikzcd}$$

in which the lower and outer rectangles are pullback, and therefore, so is the upper.

Finally, the functor $\eta^{\triangleright}_{lm}$ is given by the map $\eta^{\triangleleft}_{lm}$ for $\mathcal{E}^{\mathrm{op}}$, via Lemma 4.9. $\square$

Lemma 5.10. *Let $\mathcal{E}$ be a stringent category, and consider a pullback diagram of admissible morphisms in $\mathcal{E}$ of the following form.*

$$\begin{tikzcd} B_1 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_1 \arrow[d] \\ B_0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_0 \arrow[d] \\ 0 \arrow[r, tail] & D \end{tikzcd} \tag{5.5}$$

Then the induced map $C_1 \rightarrowtail C_0$ is an admissible monomorphism, where

$$C_i = \operatorname{coker}(B_i \rightarrowtail A_i).$$

Proof. Since $A_0 \to D$ and $A_1 \to D$ are admissible, with kernels B_0 and B_1, respectively, their unique epi-mono factorizations are through C_0, resp. C_1, and we can extend the diagram to

$$\begin{tikzcd} B_1 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_1 \arrow[rr, two heads] \arrow[d] & & C_1. \arrow[dl] \arrow[ddll, tail, bend left] \\ B_0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_0 \arrow[r, two heads] \arrow[d] & C_0 \arrow[dl, tail] & \\ 0 \arrow[r, tail] & D & & \end{tikzcd}$$

Then the statement results from the cancellation axiom. $\square$

The following result constitutes a generalization of Lemma 4.12.

Proposition 5.11. *Let $k \geq 1$, and assume that $\mathcal{E}$ is a stringent category. Then there are equivalences of simplicial categories*

$$P^{\triangleleft} S^{\langle k \rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{\langle k-1]}(\mathcal{E}),\ A \longmapsto A_{[0]\oplus -},$$
$$P^{\triangleright} S^{\langle k \rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{[k-1\rangle}(\mathcal{E}),\ A \longmapsto A_{-\oplus [0]},$$

induced by the forgetful functors. For $k \geq 2$, there is an equivalence

$$P^{\triangleleft} P^{\triangleright} S^{\langle k \rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{[k-2]}(\mathcal{E}),\ A \longmapsto A_{[0]\oplus - \oplus [0]}.$$

Proof. We prove the second statement first. For a diagram $A \in S_n^{[k-1\rangle}(\mathcal{E})$, we construct its image in $P^{\triangleright} S_n^{\langle k \rangle}(\mathcal{E}) \simeq S_{n+1}^{\langle k \rangle}(\mathcal{E})$ under the inverse functor as a right Kan extension. Namely, we extend by zero appropriately, and then into the kth dimension, as follows.

$$\begin{tikzcd} \mathrm{Fun}([k-1],[n]) \arrow[rr, "A"] \arrow[d, "\iota"'] & & \mathcal{E} \\ \mathrm{Fun}([k-1],[n+1]) \arrow[d, hook] & & \\ \mathrm{Cyl}(\iota|_{\Delta^n_{k-1}}) \arrow[uurr, dashed, "A^!"] \arrow[d, "\lambda"'] & & \\ \mathrm{Fun}([k],[n+1]) \arrow[uuurr, dashed, bend right, "\widehat{A}"'] & & \end{tikzcd} \tag{5.6}$$

Here, we have set $\iota = (d_{n+1})_*$, and the category $\mathrm{Cyl}(\iota|_{\Delta^n_{k-1}})$ is the cograph of its restriction to the skeleton. The functor λ is defined by $(s_{k-1})_*$ on Δ^n_{k-1}, and on $\mathrm{Fun}([k-1],[n+1])$, it maps

$$\alpha \mapsto \alpha \cup \{n+1\}.$$

Explicitly, $\widehat{A}$ is given by the diagram

$$\beta \longmapsto \varprojlim_{\beta \leq \lambda(\alpha)} A^!_\alpha \;\cong\; \begin{cases} A_{\beta \smallsetminus \{n+1\}} & \text{if } n+1 \in \beta, \\ \ker(A_{d_k^*\beta} \to A_{d_{k-1}^*\beta}) & \text{otherwise.} \end{cases}$$

Indeed, $d_k^*\beta$ is initial amongst those objects of the indexing category of the limit which come from $\mathrm{Fun}([k-1],[n+1])$. If $n+1 \in \beta$, then this is the only contribution. Otherwise, there are additionally the objects of the form

$$[\alpha] \in \Delta^n_{k-1} \text{ with } d_{k-1}^*\beta \leq \alpha.$$

Therefore, in that case, the limit reduces to just the pullback

$$\widehat{A}_\beta \cong \varprojlim \left[\begin{tikzcd} & A^!_{d_k^*\beta} \arrow[d] \\ A^!_{[d_{k-1}^*\beta]} \arrow[r] & A^!_{d_{k-1}^*\beta} \end{tikzcd} \right] \cong \varprojlim \left[\begin{tikzcd} & A_{d_k^*\beta} \arrow[d] \\ 0 \arrow[r] & A_{d_{k-1}^*\beta} \end{tikzcd} \right] = \ker(A_{d_k^*\beta} \to A_{d_{k-1}^*\beta}).$$

Now let γ be a $(k+1)$-simplex in Δ^{n+1}. We claim that the corresponding sequence $\widehat{A}_{d_\bullet^*\gamma}$ is exact. If $n+1 \in \gamma$, then this is simply given by

$$\ker(A_{d_k^* d_{k+1}^*\gamma} \to A_{d_{k-1}^* d_{k+1}^*\gamma}) \rightarrowtail A_{d_k^*\gamma \smallsetminus \{n+1\}} \longrightarrow \dots \longrightarrow A_{d_1^*\gamma \smallsetminus \{n+1\}} \twoheadrightarrow A_{d_0^*\gamma \smallsetminus \{n+1\}}$$

which is an exact sequence in $\mathcal{E}$ by definition.

Otherwise, the relevant sequence is given by

$$
\begin{tikzcd}
\ker(A_{d_k^* d_{k+1}^* \gamma} \to A_{d_{k-1}^* d_{k+1}^* \gamma}) \arrow[d] & & \ker(A_{d_k^* d_{k+1}^* \gamma} \to A_{d_{k-1}^* d_{k+1}^* \gamma}) \arrow[d] \\
\ker(A_{d_k^* d_k^* \gamma} \to A_{d_{k-1}^* d_k^* \gamma}) \arrow[d] & & \ker(A_{d_k^* d_{k+1}^* \gamma} \to A_{d_{k-1}^* d_k^* \gamma}) \arrow[d] \\
\ker(A_{d_k^* d_{k-1}^* \gamma} \to A_{d_{k-1}^* d_{k-1}^* \gamma}) \arrow[d] & \text{or equivalently,} & \ker(A_{d_{k-1}^* d_{k+1}^* \gamma} \to A_{d_{k-1}^* d_k^* \gamma}) \arrow[d] \\
\ker(A_{d_k^* d_{k-2}^* \gamma} \to A_{d_{k-1}^* d_{k-2}^* \gamma}) \arrow[d] & & \ker(A_{d_{k-2}^* d_{k+1}^* \gamma} \to A_{d_{k-2}^* d_k^* \gamma}) \arrow[d] \\
\vdots \arrow[d] & & \vdots \arrow[d] \\
\ker(A_{d_k^* d_0^* \gamma} \to A_{d_{k-1}^* d_0^* \gamma}) & & \ker(A_{d_0^* d_{k+1}^* \gamma} \to A_{d_0^* d_k^* \gamma}).
\end{tikzcd}
$$

We prove exactness inductively. The first part of the sequence fits into a diagram of the form

$$
\begin{tikzcd}
\widehat{A}_{d_{k+1}^* \gamma} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widehat{A}_{d_k^* \gamma} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_{d_k^* d_{k+1}^* \gamma} \arrow[d] \\
0 \arrow[r, tail] & \widehat{A}_{d_{k-1}^* \gamma} \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_{d_{k-1}^* d_{k+1}^* \gamma} \arrow[d] \\
 & 0 \arrow[r, tail] & A_{d_{k-1}^* d_k^* \gamma}.
\end{tikzcd}
$$

By definition, the bottom left square is pullback, so we can pull it back to the top and then to the left, since each outer rectangle is a pullback square by construction. Thus,

$$
\widehat{A}_{d_{k+1}^* \gamma} \rightarrowtail \widehat{A}_{d_k^* \gamma} \longrightarrow \widehat{A}_{d_{k-1}^* \gamma}
$$

is a left exact sequence. Now, for each $0 < i < k$, let us write

$$
\begin{tikzcd}
(A_{d_{i+1}^* d_{k+1}^* \gamma} \twoheadrightarrow)\ B_{i+1} \arrow[r, tail] \arrow[d, dashed] & A_{d_i^* d_{k+1}^* \gamma} \arrow[r, two heads] \arrow[d, dashed] & B_i\ (\rightarrowtail A_{d_{i-1}^* d_{k+1}^* \gamma}) \arrow[d, dashed] \\
(A_{d_{i+1}^* d_k^* \gamma} \twoheadrightarrow)\ C_{i+1} \arrow[r, tail] & A_{d_i^* d_k^* \gamma} \arrow[r, two heads] & C_i\ (\rightarrowtail A_{d_{i-1}^* d_k^* \gamma})
\end{tikzcd}
\tag{5.7}
$$

for the corresponding short exact sequences at the ith vertex of $d_{k+1}^* \gamma$ and $d_k^* \gamma$, respectively. First, we show that $B_k \to C_k$ is an admissible epimorphism. But we have

$$
B_k = \operatorname{coker}(\widehat{A}_{d_{k+1}^* \gamma} \rightarrowtail A_{d_k^* d_{k+1}^* \gamma}), \text{ and } C_k = \operatorname{coker}(\widehat{A}_{d_k^* \gamma} \rightarrowtail A_{d_k^* d_k^* \gamma}).
$$

Therefore, they fit into a diagram of the following form, which yields the claim.

$$
\begin{tikzcd}
\widehat{A}_{d_{k+1}^* \gamma} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & \widehat{A}_{d_k^* \gamma} \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & A_{d_k^* d_{k+1}^* \gamma} \arrow[d, two heads] \\
0 \arrow[r, tail] & B_k' \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & B_k \arrow[d, two heads] \\
 & 0 \arrow[r, tail] & C_k
\end{tikzcd}
$$

In particular, $B'_k = \ker(B_k \twoheadrightarrow C_k)$. Next, we show that $B_{k-1} \to C_{k-1}$ admits a kernel B'_{k-1} in $\mathcal{E}$. In fact, consider the following diagram.

$$\begin{tikzcd}
B_k \arrow[r, tail] \arrow[d, two heads] & A_{d^*_{k-1}d^*_{k+1}\gamma} \arrow[r, two heads] \arrow[d, two heads] & B_{k-1} \arrow[d, dashed, two heads, "(3)"] \\
C_k \arrow[r, dashed, tail, "(1)"] \arrow[d, "="'] & E \arrow[r, dashed, two heads, "(1')"] \arrow[d, tail] & D \arrow[d, dashed, tail, "(2)"] \\
C_k \arrow[r, tail] \arrow[d, two heads] & A_{d^*_{k-1}d^*_k\gamma} \arrow[r, two heads] \arrow[d, two heads] & C_{k-1} \arrow[d, dashed, two heads, "(2')"] \\
0 \arrow[r, tail] & C'_{k-1} \arrow[r, "="] & C'_{k-1}
\end{tikzcd}$$

We have (1) by Remark 5.4, and (1′) is its cokernel. The snake lemma yields (2) and (2′), and (3) is obtained dually to (1).

Now the snake lemma implies that the top row of the following diagram is short exact.

$$\begin{tikzcd}
B'_k \arrow[r, tail] \arrow[d, tail] & \widehat{A}_{d^*_{k-1}\gamma} \arrow[r] \arrow[d, tail] & B'_{k-1} \arrow[d, tail] \\
B_k \arrow[r, tail] \arrow[d, two heads] & A_{d^*_{k-1}d^*_{k+1}\gamma} \arrow[r, two heads] \arrow[d] & B_{k-1} \arrow[d] \\
C_k \arrow[r, tail] & A_{d^*_{k-1}d^*_k\gamma} \arrow[r, two heads] & C_{k-1}
\end{tikzcd}$$

In particular, this settles the case $k = 2$. For $k \geq 3$, we can rewrite the diagram

$$\begin{tikzcd}
B_2 \arrow[r, tail] \arrow[d] & A_{d^*_1 d^*_{k+1}\gamma} \arrow[r, two heads] \arrow[d] & B_1 = A_{d^*_0 d^*_{k+1}\gamma} \arrow[d] \\
C_2 \arrow[r, tail] & A_{d^*_1 d^*_k\gamma} \arrow[r, two heads] & C_1 = A_{d^*_0 d^*_k\gamma}
\end{tikzcd}$$

in terms of the hyperplane lemma (Proposition 5.9), namely as the upper part of the short exact sequence of acyclic sequences

$$\begin{tikzcd}
\eta^{\triangleright}_{(n-\gamma_1)(n-\gamma_0)}(A)_{d^*_{k-1}\alpha} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0\}\cup d^*_{k-1}\alpha} \arrow[r, two heads] \arrow[d] & A_{\{\gamma_1\}\cup d^*_{k-1}\alpha} \arrow[d] \\
\eta^{\triangleright}_{(n-\gamma_1)(n-\gamma_0)}(A)_{d^*_{k-2}\alpha} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0\}\cup d^*_{k-2}\alpha} \arrow[r, two heads] \arrow[d] & A_{\{\gamma_1\}\cup d^*_{k-2}\alpha} \arrow[d] \\
\eta^{\triangleright}_{(n-\gamma_1)(n-\gamma_0)}(A)_{d^*_{k-3}\alpha} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0\}\cup d^*_{k-3}\alpha} \arrow[r, two heads] \arrow[d] & A_{\{\gamma_1\}\cup d^*_{k-3}\alpha} \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd} \tag{5.8}$$

where $\alpha = d^*_0 d^*_1\gamma$. In particular, $B_2 \to C_2$ is an admissible morphism. Applying the snake lemma to the third morphism of short exact sequences in (5.8) tells us that the map

$$C'_2 = \operatorname{coker}(B_2 \to C_2) \longrightarrow \operatorname{coker}(A_{d^*_1 d^*_{k+1}\gamma} \to A_{d^*_1 d^*_k\gamma})$$

is an admissible monomorphism, and therefore, by applying it to the first, that $\widehat{A}_{d^*_1\gamma} \twoheadrightarrow \widehat{A}_{d^*_0\gamma}$.

Finally, we can iterate the argument, realizing (5.7) as the upper part of the diagram

$$\begin{tikzcd}
\eta^{(i)}(A)_{d^*_{k-i}\alpha^{(i)}} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0,\dots,\gamma_{i-1}\}\cup d^*_{k-i}\alpha^{(i)}} \arrow[r, two heads] \arrow[d] & \eta^{(i-1)}(A)_{d^*_{k-i+1}\alpha^{(i-1)}} \arrow[d] \\
\eta^{(i)}(A)_{d^*_{k-i-1}\alpha^{(i)}} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0,\dots,\gamma_{i-1}\}\cup d^*_{k-i-1}\alpha^{(i)}} \arrow[r, two heads] \arrow[d] & \eta^{(i-1)}(A)_{d^*_{k-i}\alpha^{(i-1)}} \arrow[d] \\
\eta^{(i)}(A)_{d^*_{k-i-2}\alpha^{(i)}} \arrow[r, tail] \arrow[d] & A_{\{\gamma_0,\dots,\gamma_{i-1}\}\cup d^*_{k-i-2}\alpha^{(i)}} \arrow[r, two heads] \arrow[d] & \eta^{(i-1)}(A)_{d^*_{k-i-1}\alpha^{(i-1)}} \arrow[d] \\
\vdots & \vdots & \vdots
\end{tikzcd}$$

where $\eta^{(i)} = \eta^{\triangleright}_{(n-\gamma_i)(n-\gamma_{i-1})} \circ \dots \circ \eta^{\triangleright}_{(n-\gamma_1)(n-\gamma_0)}$ and $\alpha^{(i)} = d_0^* \dots d_i^* \gamma$. Then the sequence

$$B'_{i+1} \rightarrowtail \widehat{A}_{d_i^*\gamma} \longrightarrow B'_i$$

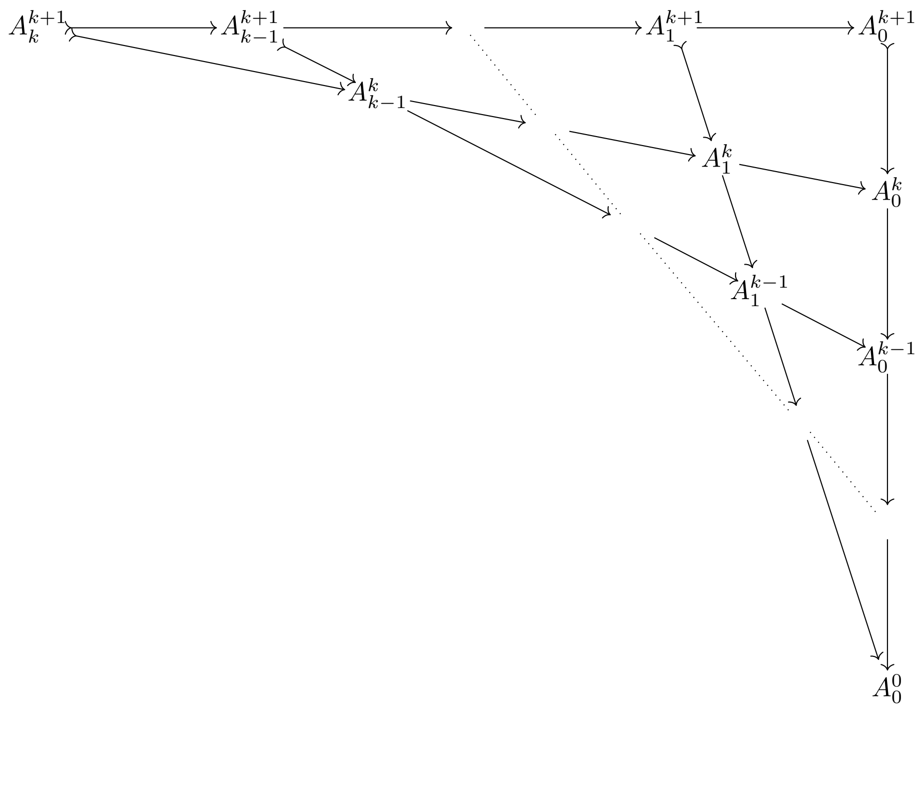
is the beginning of the corresponding long exact snake, where $B'_i = \ker(B_i \to C_i)$, and is therefore a short exact sequence, as above.

Finally, the equivalence $P^{\triangleleft} S^{\langle k \rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{\langle k-1]}(\mathcal{E})$ follows via Lemma 4.9 from the one we have proven above. Furthermore, if $k \geq 2$, we obtain $P^{\triangleleft} P^{\triangleright} S^{\langle k \rangle}(\mathcal{E}) \xrightarrow{\simeq} S^{[k-2]}(\mathcal{E})$ as an immediate consequence of the two. Namely, let $A \in S_n^{[k-2]}(\mathcal{E})$. Then the left Kan extension analogous to (5.6) produces a diagram

$$\widehat{A} \in S_{n+1}^{[k-1\rangle}(\mathcal{E}) = P^{\triangleleft} S_n^{[k-1\rangle}(\mathcal{E}) \simeq P^{\triangleleft} P^{\triangleright} S_n^{\langle k \rangle}(\mathcal{E}),$$

as all arguments above apply verbatim to show that $\widehat{A}$ consists of right exact sequences. $\square$

Remark 5.12. Proposition 5.11 can be seen as a higher analogue of the third isomorphism theorem, in that the equivalence of categories $S_{k+1}^{\langle k-1]}(\mathcal{E}) \xrightarrow{\simeq} P^{\triangleleft} S_{k+1}^{\langle k \rangle}(\mathcal{E}) = S_{k+2}^{\langle k \rangle}(\mathcal{E})$ boils down to the following statement. Given a configuration of left exact sequences of the form



where $A_i^j = A_{d_i^* d_j^* \Delta^{k+1}_{k+1}}$ in the previous notation, the induced maps between the cokernels

$$\operatorname{coker}(A_1^{k+1} \to A_0^{k+1}) \longrightarrow \operatorname{coker}(A_1^k \to A_0^k) \longrightarrow \dots \longrightarrow \operatorname{coker}(A_1^0 \to A_0^0)$$

constitute an exact sequence in $\mathcal{E}$.

Remark 5.13. Let us use the observation in Remark 5.12 to illustrate that the stringency assumption on $\mathcal{E}$ in Proposition 5.11 is necessary. For this, consider the dual situation of Remark 5.7, so that both f and the composition $A \xrightarrow{f} B \xrightarrow{g} C$ are admissible epimorphisms but g is not. Now let $B' \rightarrowtail A$, resp. $C' \rightarrowtail A$, be the kernel of f, resp. $g \circ f$. If $h\colon B' \to C'$ is admissible,

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B' \arrow[d, "h"] \\
0 \arrow[r, tail] & 0 \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & C' \arrow[d] \\
 & 0 \arrow[r, tail] & A
\end{tikzcd}$$

lies in $S_3^{\langle 1]}(\mathcal{E})$, but it cannot be extended to $P^{\triangleleft} S_3^{\langle 2 \rangle}(\mathcal{E})$ since g is not admissible. If h itself is not admissible either but admits a kernel D, we can construct a similar element

$$\begin{tikzcd}
D \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & D \oplus B' \arrow[d, "\begin{pmatrix} 0 & 1 \\ 0 & h \end{pmatrix}"] \\
0 \arrow[r, tail] & B' \arrow[r, tail, "h"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B' \oplus C' \arrow[d, "(0{,}1)"] \\
 & 0 \arrow[r, tail] & A
\end{tikzcd}$$

of $S_3^{\langle 1]}(\mathcal{E})$. The corresponding sequence of cokernels $B' \xrightarrow{h} C' \to B \xrightarrow{g} C$ is again not exact.

Corollary 5.14. *If $\mathcal{E}$ is stringent, there are natural equivalences of simplicial categories*

$$\begin{aligned}
S^{\langle k]}(\mathcal{E}) &\xrightarrow{\simeq} T(S^{\langle 0]} S^{\langle k \rangle}(\mathcal{E})),\\
S^{[k \rangle}(\mathcal{E}) &\xrightarrow{\simeq} T(S^{\langle k \rangle} S^{[0 \rangle}(\mathcal{E})),\\
S^{[k]}(\mathcal{E}) &\xrightarrow{\simeq} T(S^{\langle 0]} S^{\langle k \rangle} S^{[0 \rangle}(\mathcal{E})).
\end{aligned}$$

Proof. This is an immediate consequence of Proposition 5.11 and Theorem 4.15, as

$$S^{\langle k]}(\mathcal{E}) \simeq P^{\triangleleft} S^{\langle k+1 \rangle}(\mathcal{E}) \simeq P^{\triangleleft} T S^{\langle k+1 \rangle}(\mathcal{E}) \simeq T P^{\triangleleft} S^{\langle k+1 \rangle}(\mathcal{E}) \simeq T(S^{\langle 0]} S^{\langle k \rangle}(\mathcal{E})),$$

by Lemma 2.18. □

Theorem 5.15. *Let $\mathcal{E}$ be a proto-exact category. The two-dimensional Waldhausen construction $S^{\langle 2 \rangle}(\mathcal{E})$ is an upper 3-Segal category if and only if $\mathcal{E}$ is proto-abelian.*

Proof. By the path space criterion, it suffices to show that $P^{\triangleleft} P^{\triangleright} S^{\langle 2 \rangle}(\mathcal{E})$ is Segal. But Proposition 5.11 below shows that for all $n \geq 2$, the forgetful functor

$$P^{\triangleleft} P^{\triangleright} S^{\langle 2 \rangle}_{n-2}(\mathcal{E}) \longrightarrow S^{[0]}_{n-2}(\mathcal{E}),\ A \longmapsto (A_{01n} \to A_{02n} \to \dots \to A_{0(n-1)n}),$$

is an equivalence of categories, identifying the double path space $P^{\triangleleft} P^{\triangleright} S^{\langle 2 \rangle}(\mathcal{E}) \xrightarrow{\simeq} N^{\mathcal{E}}(\mathcal{E})$ with the categorified nerve of $\mathcal{E}$, cf. Example 4.7 (1).

Conversely, the Segal condition for $P^{\triangleleft} P^{\triangleright} S^{\langle 2 \rangle}(\mathcal{E}) \simeq S^{[0]}(\mathcal{E})$ requires admissible morphisms in $\mathcal{E}$ be closed under composition, hence $\mathcal{E}$ must be proto-abelian already. □

Remark 5.16. Suppose $\mathcal{E}$ is additive. Then Theorem 5.15 does not generalize to the higher dimensional Waldhausen constructions, that is, $S^{\langle k\rangle}(\mathcal{E})$ is not upper $(2k-1)$-Segal for $k \neq 2$. Indeed, the diagram

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A \arrow[r, "="] \arrow[d, "{(0,1)}"] & A \arrow[d, "="] \\
 & 0 \arrow[r] & A \arrow[r, "{(1,0)}"] \arrow[d] & A \oplus A \arrow[r, "{(1,1)}"'] \arrow[d, "{(0,1)}"'] & A \arrow[d] \\
 & & 0 \arrow[r] & A \arrow[r] \arrow[d] & 0 \arrow[d] \\
 & & & 0 \arrow[r] & 0 \arrow[d] \\
 & & & & 0
\end{tikzcd} \tag{5.9}$$

is an element of the right-hand side of the lower 3-Segal map for $P^{\lhd}P^{\rhd}S_4^{\langle 3\rangle}(\mathcal{E})$, but does not lie in its essential image.

We are now prepared to prove our main result. In particular, by Proposition 5.11 as well as Lemma 4.9, the path space criterion provides a new proof of Theorem 4.13, under the additional assumption that $\mathcal{E}$ be stringent.

Proof of Proposition 5.6. Let $n \geq 2k$. We show inductively that the lower $(2k-1)$-Segal map

$$S_n^{\langle k-1]}(\mathcal{E}) \longrightarrow \varprojlim_{I \in \mathcal{L}([n],2k-1)} S_I^{\langle k-1]}(\mathcal{E}) \tag{5.10}$$

is an equivalence. Throughout, for $0 \leq i \leq n$, let δ_i refer to the ith face map of Δ_n^n, even when applied to any subsimplex of it. That is, δ_i^* removes the vertex i, and $\widehat{\delta}_i^*$ adjoins it.

First, consider the case $n = 2k$. By Lemma 5.17, the only k-simplex in Δ^{2k} not contained in an even subset of $[2k]$ of cardinality $2k$ already is

$$\varepsilon = \{0, 2, \ldots, 2k\} = \delta_{2k-1}^*\delta_{2k-3}^* \cdots \delta_1^*\Delta_{2k}^{2k}.$$

But if A lies in the right-hand side of (5.10), then we can form the unique compositions

$$\begin{tikzcd}[column sep=small]
A_{\delta_{2k}^*\varepsilon} \arrow[rr, dashed] \arrow[dr, tail] & & A_{\delta_{2k-2}^*\varepsilon} \arrow[rr, dashed] \arrow[dr] & & \ldots \arrow[rr, dashed] \arrow[dr] & & A_{\delta_0^*\varepsilon} \\
 & A_{\delta_{2k}^*\widehat{\delta}_{2k-1}^*\delta_{2k-2}^*\varepsilon} \arrow[ur, tail] & & A_{\delta_{2k-2}^*\widehat{\delta}_{2k-3}^*\delta_{2k-4}^*\varepsilon} \arrow[ur] & \ldots & A_{\delta_2^*\widehat{\delta}_1^*\delta_0^*\varepsilon} \arrow[ur] &
\end{tikzcd}$$

completing A to an element of $S_{2k}^{\langle k-1]}(\mathcal{E})$. It remains to be shown that the resulting sequence

$$A_{\delta_{2k}^*\varepsilon} \longrightarrow A_{\delta_{2k-2}^*\varepsilon} \longrightarrow \ldots \longrightarrow A_{\delta_0^*\varepsilon}$$

is left exact. We proceed by induction. The case $k = 2$ is settled by Theorem 5.15. In general, since $A_{\delta_{2k}^*\varepsilon} \rightarrowtail A_{\delta_{2k-2}^*\varepsilon}$ is an admissible monomorphism (as a composition of such), it suffices to prove that

$$\operatorname{coker}(A_{\delta_{2k}^*\varepsilon} \rightarrowtail A_{\delta_{2k-2}^*\varepsilon}) \longrightarrow A_{\delta_{2k-4}^*\varepsilon} \longrightarrow \ldots \longrightarrow A_{\delta_0^*\varepsilon} \tag{5.11}$$

is a left exact sequence. For this, we use the hyperplane lemma. Namely, the functor

$$\eta_{(2k-2)2k}^{\lhd} \colon S_{2k}^{\langle k-1]}(\mathcal{E}) \longrightarrow S_{2k-2}^{\langle k-2]}(\mathcal{E})$$

constructed in Proposition 5.9 is compatible with the corresponding lower Segal maps on both sides, in that it induces a commutative diagram of the following form.

$$\begin{tikzcd} S_{2k}^{\langle k-1]}(\mathcal{E}) \arrow[r] \arrow[d, "\eta^{\triangleleft}_{(2k-2)2k}"] & \varprojlim_{I \in \mathcal{L}([2k],2k-1)} S_I^{\langle k-1]}(\mathcal{E}) \arrow[d, dashed, "\eta^{\triangleleft}_{(2k-2)2k}"] \\ S_{2k-2}^{\langle k-2]}(\mathcal{E}) \arrow[r] & \varprojlim_{J \in \mathcal{L}([2k-2],2k-3)} S_J^{\langle k-2]}(\mathcal{E}) \end{tikzcd}$$

Indeed, this is because we have $d_0^* d_0^* \varepsilon = \delta_{2k-3}^* \delta_{2k-5}^* \cdots \delta_1^* \Delta_{2k-2}^{2k-2}$. But then, by induction, the lower horizontal map is an equivalence, which by the above means precisely that (5.11) is a left exact sequence.

In order to prove the Segal conditions for the higher cells $S_n^{\langle k-1]}(\mathcal{E})$, we once again employ induction, now on the dimension n. If A lies in the right-hand side of the lower $(2k-1)$-Segal map for $S_n^{\langle k-1]}(\mathcal{E})$, we first need to see that taking compositions completes A to a well-defined diagram of shape $\mathrm{Fun}([k-1],[n])$ in $\mathcal{E}$.

By Lemma 5.17, we need only consider sequences indexed by simplices γ with all vertices separated by gaps. For gaps $i \in [n]$ of size 1, there is (as before) a unique composition,

$$\begin{tikzcd} A_{\delta_{i+1}^*\gamma} \arrow[rr, dashed] \arrow[dr] & & A_{\delta_{i-1}^*\gamma}. \\ & A_{\delta_{i+1}^*\widehat{\delta}_i^*\delta_{i-1}^*\gamma} \arrow[ur] & \end{tikzcd}$$

For a gap of γ of size $l+1$, say $\{i,\ldots,i+l\} \subseteq [n]$, each $0 \le j \le l$ defines the composition

$$\begin{tikzcd} A_{\delta_{i+l+1}^*\gamma} \arrow[rr, dashed] \arrow[dr] & & A_{\delta_{i-1}^*\gamma}. \\ & A_{\delta_{i+l+1}^*\widehat{\delta}_{i+j}^*\delta_{i-1}^*\gamma} \arrow[ur] & \end{tikzcd}$$

By induction, all possible compositions can be reduced to one of these. On the other hand, they all agree, since for all $0 \le j < j' \le l$, the following diagram commutes.

$$\begin{tikzcd} A_{\delta_{i+l+1}^*\gamma} \arrow[r] \arrow[d] & A_{\delta_{i+l+1}^*\widehat{\delta}_{i+j'}^*\delta_{i-1}^*\gamma} \arrow[d] \\ A_{\delta_{i+l+1}^*\widehat{\delta}_{i+j}^*\delta_{i-1}^*\gamma} \arrow[r] \arrow[ur] & A_{\delta_{i-1}^*\gamma} \end{tikzcd}$$

Finally, we apply induction to obtain the remaining exactness conditions for the completed diagram of A. Namely, the sequence indexed by γ is left exact, since $\partial_i(A)$ lies in the right-hand side of the lower $(2k-1)$-Segal map of $S_{n-1}^{\langle k-1]}(\mathcal{E})$, for any gap i of γ. $\square$

Lemma 5.17. *Let $n \ge 2k$. Let γ be a k-subsimplex of Δ^n with a pair of adjacent simplices. Then γ is contained in an even subset $I \subseteq [n]$ of cardinality $\#I = 2k$.*

Proof. For $n = 2k$, this is clear. For the induction step, we can assume that $n \in \gamma$, otherwise the statement follows tautologically from the induction hypothesis. Let $0 < m < n$ be the maximal gap of γ. By induction, $(\gamma \cup \{m\}) \smallsetminus \{n\}$ is contained in an even subset $I' \subseteq [n-1]$ with $\#I' = 2k$. But then γ is contained in $I = (I' \smallsetminus \{m\}) \cup \{n\}$, which is even in $[n]$. $\square$

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